

# The Monte-Carlo tools for the Drell-Yan production of W and Z bosons for the precision measurements era

*Collinear Factorization and parton shower Monte Carlo*

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More on <http://jadach.web.cern.ch/>



Parton shower MC (psMC) is a workhorse  
in the data analysis in hadron colliders

## THE CHALLENGE:

Can we put more of pQCD into  
parton shower MCs?  
And even PDFs?



# QCD and pQCD

Last ~40 years

- **1954–75:** establishing QCD as renormalizable QFT with asymptotic freedom
- **1976–83:** "Gold-rush epoch". Principles of practical pQCD calculations; CFTs, PDFs, NLO at  $e^+e^-$ , DIS, DY (but experiment 20 years behind theory!).
- **1984-2002:** Experimental tests at PEP/PETRA, Tevatron, LEP and consolidation of relevant th. tools.
- **2002-Now:** Important/crucial role of QCD – helping data analysis at hadron colliders. Extending/refining practical toolbox of pQCD.



# pQCD and Monte Carlo

- **1974-84:** parton shower MC (psMC) simulating hadronization only.
- **1984/85:** psMC simulates LO/LL evolution of the parton distributions (DGLAP).
- **1976-Now:** MC used to integrate tree-level multiparton phase space, recently including NLO corrections to hard process.
- **2002-Now:** Methods of combining LO psMC with LO+NLO hard process (MC@NLO...)
- **1985-Now:** psMC stays at LO/LL level; Why?????!!!!

# Why psMC stays at LO since 1985?

**Possible answers before  $\sim$ 2004:**

- Not important/urgent (poor exp. data).
- Unfeasible, or at least extremely difficult.

**Both above answers are now invalid:**

- LHC data analysis will require improved QCD, NLO psMC in particular.
- Krakow group shows that NLO psMC is feasible:-) Albeit difficult:-(



# Krakow (IFJPAN) R&D activity in pQCD, 2004-now

- **2004-2007:** Constructing new variants of LO parton shower MC: Constrained MC algorithm, CCFM, inclusive NLO kernels, first studied on YFS-style inclusion NLO corrs. to hard process.
- **2007-2010:** Constructing NLO parton shower MC: re-calculating NLO kernels (non-singlet), working out 2 scenarios for including NLO corrections in exclusive form, studies on soft non-abelian limit of exclusive LO+NLO kernels, studies of factorization scheme dependence
- **2010-2011:** New YFS-like schemes of including NLO in hard process ME in W/Z production at LHC and DIS at HERA (alternative to MC@NLO) and in the ladder/uPDF.

S-J, K.Golec-Biernat, A.Kusina, M.Skrzypek, W.Placzek, M.Slawinska, P.Stephens, P.Stoklosa, Z.Wąs.



# Main lessons from Krakow R&D effort

- Backward evolution MC algorithm for initial state parton shower is not the only solution (important technical point)
- QCD Collinear Factorization Theorems (CFTs) to be reorganized before pushing psMC beyond LO:
  - (a) Violation of 4-mom. to be repaired
  - (b) Explicit time-ordered exponent in the phase space,
- Soft gluon limit beyond LO dictates angular ord.
- LO psMC has to be reorganized before NLO included.
- Inevitable departure from standard  $\overline{MS}$ , CFP scheme:  
extra term  $\alpha_S P(z) \ln(1-z)$  in coeff. function  $C_2$  and  
 $\alpha_S \int_{z=z_1 z_2} P(z_1) P(z_2) \ln(z_1(1-z_2)/(1-z_1))$  in NLO kernels.
- New YFS-like scheme for NLO cors. to hard process – an attractive alternative to MC@NLO and POWHEG?



# Even more...

**psMC can absorb many objects and techniques of pQCD presently in non-MC (or analytical) world:**

- **Universal PDFs, their definition,  $\ln(Q)$  evolution, and even extraction from the data,**
- **kT factorization/resummation,**
- **low-x modelling (CCFM, BFKL),**
- **finite quark mass thresholds,**
- **soft gluon resummations of  $\frac{\ln^n(1-x)}{(1-x)}$  etc., both in the hard process and the ladder.**



# Main OBSTACLE on the way to new psMC: Collinear Factorization Theorems (CFTs)

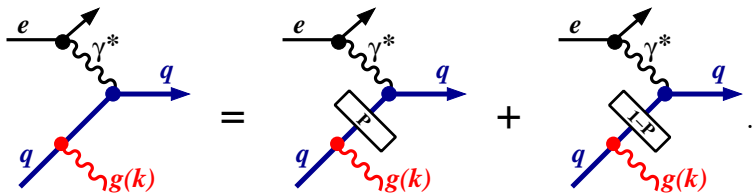
- **Classic CFTs** were formulated by:
  - △ 1979, EGMPR [Ellis, Georgi, Machacek, Politzer, Ross],
  - △ 1980, CFP [Curci, Furmanski, Petronzio],
  - △ 1980-85, CSS [Collins, Stermann, Soper, Bodwin,...]
- **Why not suited** for the parton shower MC?
  - ★ **Non-conservation of the 4-momenta**
  - ★ **Over-subtractions**
- **How to modify CFT?** for use in psMC:
  - Redefine **projection operators** for extracting singular (Coll.) parts, such that 4-mom. is conserved.
  - Introduce **time-ordered exponential** earlier, while isolating/subtracting Coll/IR singular parts.



# Non-conservation of 4-momentum in CFTs

The culprit is LO-extracting “casting” operator  $\mathbb{P}$  of CFTs

$$\text{Exact} = \text{LO} + \text{NLO}$$



$$\int_{m_p}^{\text{TruePhSp.}} \frac{dk_T}{k_T} (\dots) = \int dk_T (\dots) \delta(k_T) \int_{m_p}^{\mu_F} \frac{dk'_T}{k'_T} + \dots$$

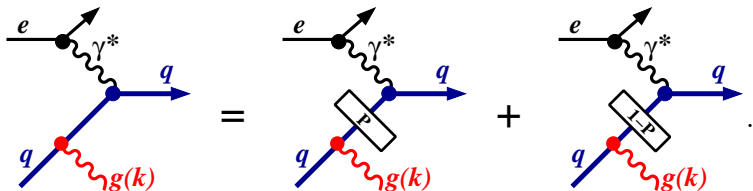
$\int dk_T$  above and below  $\mathbb{P}$  decouples.



# Non-conservation of 4-momentum in CFTs

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$$\int_{m_p}^{\text{TruePhSp.}} \frac{dk_T}{k_T} (\dots) = \int dk_T (\dots) \delta(k_T) \ln \frac{\mu_F}{m_p} + \dots$$

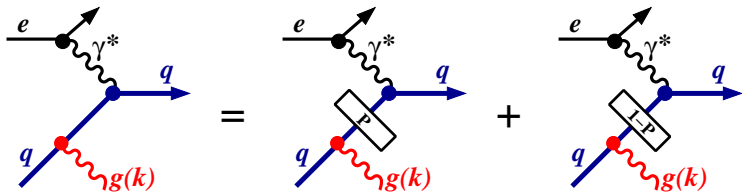
Explicit collinear big logarithm.



# Non-conservation of 4-momentum in CFTs

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$$\text{Exact} = \text{LO} + \text{NLO}$$



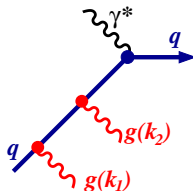
$$\int_0^{\text{PhSp.}} \frac{dk_T}{k_T} \left( \frac{k_T}{\mu_F} \right)^\varepsilon = \frac{1}{\varepsilon} + (1 - PP) \int_0^{\text{PhSp.}} \frac{dk_T}{k_T} \left( \frac{k_T}{\mu_F} \right)^\varepsilon$$

Dimensional regularization obscures the picture;  $\delta(k_T)$  implicit.

# Over-subtractions, over-cancellations in CFTs

## The real disaster strikes in case of 2 gluons!

True result from Feynman diagram is:



A Feynman diagram showing a blue quark line. The line starts from the bottom left, goes up and right to a red dot, then continues up and right to another red dot, and finally goes right to a blue dot. From the first red dot, a red wavy gluon line labeled  $g(k_1)$  goes down and right. From the second red dot, a red wavy gluon line labeled  $g(k_2)$  goes down and right. From the blue dot, a black wavy photon line labeled  $\gamma^*$  goes up and left. A blue arrow labeled  $q$  points to the right from the blue dot.

$$\rightarrow \frac{1}{2} \ln^2 \frac{\mu_F}{m_p}$$

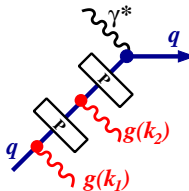


# Over-subtractions, over-cancellations in CFTs

## The real disaster strikes for 2 gluons!

**Apparently wrong LO!** In addition to  $k_T$  non-conservation:

Double use of casting operator yields wrong LO:



The diagram shows a quark line (blue) with momentum  $q$  entering from the bottom left. It splits into a quark (momentum  $q$ ) and a gluon (momentum  $g(k_1)$ ). The gluon splits into a gluon (momentum  $g(k_2)$ ) and a photon (momentum  $\gamma^*$ ). The photon splits into a quark (momentum  $q$ ) and a gluon (momentum  $g(k_2)$ ). The quark and gluon from the second splitting are shown as a jet. Three rectangular boxes labeled 'P' are placed along the quark line, representing casting operators. An arrow points from the diagram to the following equation.

$$\rightarrow 1 \ln^2 \frac{\mu_F}{m_p}$$

**WAIT! CFTs self-correct for the above LO over-estimate!**

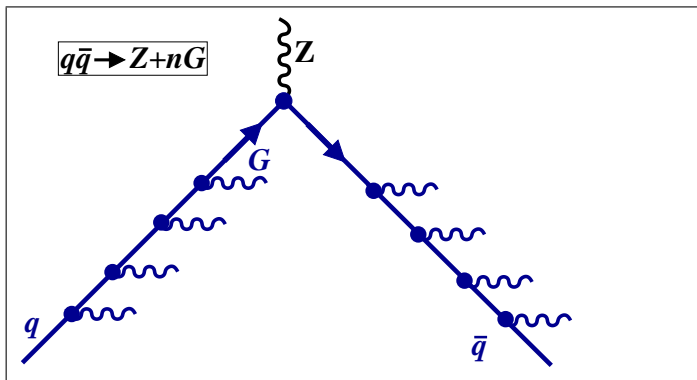
– But in a way which inhibits MC implementation of CFTs.

# Lets go to full size CFT...



# EGMPR factorization theorem/scheme

R. K. Ellis, H. Georgi, M. Machacek, H. D. Politzer and G. G. Ross,  
Nucl. Phys. B 152, 285 (1979). See also J. Collins Phys.Rev. D58, 1998

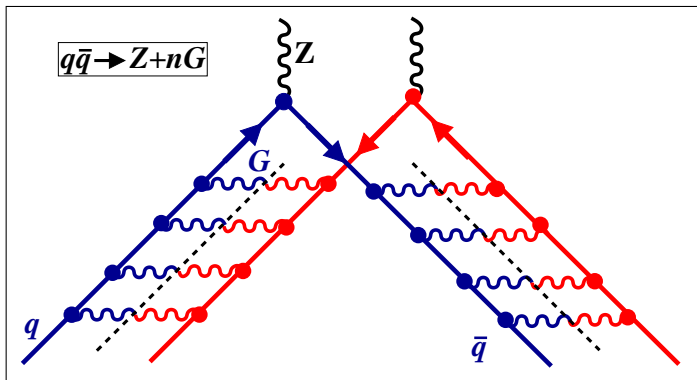


Gluostrahlung diagram



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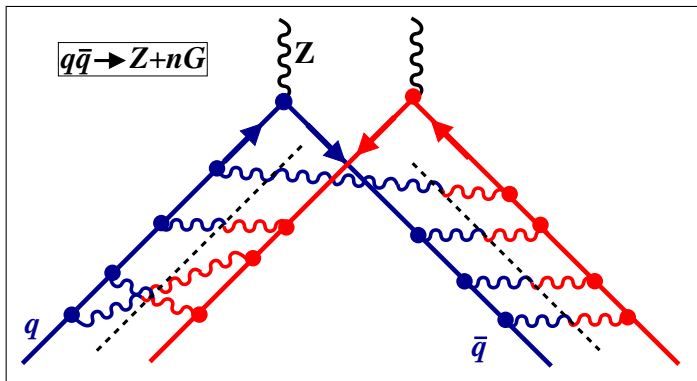


Gluostrahlung diagram **squared**



# EGMPR factorization theorem/scheme

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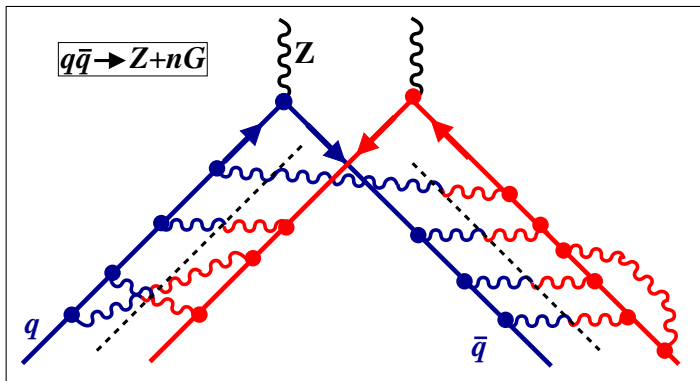


Gluostrahlung diagram squared +interferences



# EGMPR factorization theorem/scheme

R. K. Ellis, H. Georgi, M. Machacek, H. D. Politzer and G. G. Ross,  
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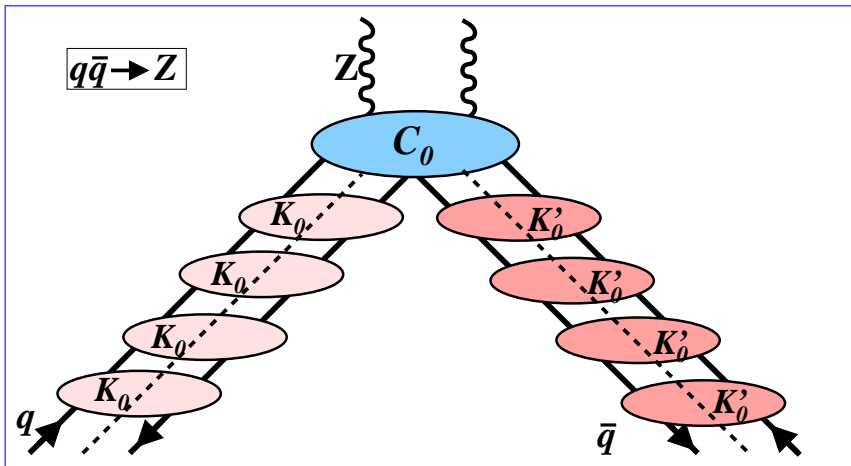


Gluostrahlung diagram squared +interferences +virtuals



# EGMPR **raw** factorization theorem

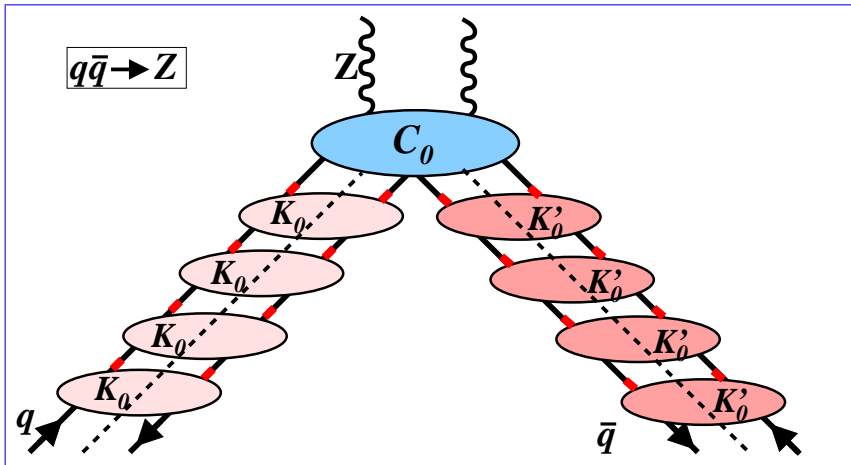
Infinite order analysis if coll. singularities in axial gauge



- $C_0$  is finite,  $K_0, K'_0$  are 2PI kernel. Phase space intact!
- Vertical rungs **I** are source of ALL Coll. singularities.

# EGMPR **raw** factorization theorem

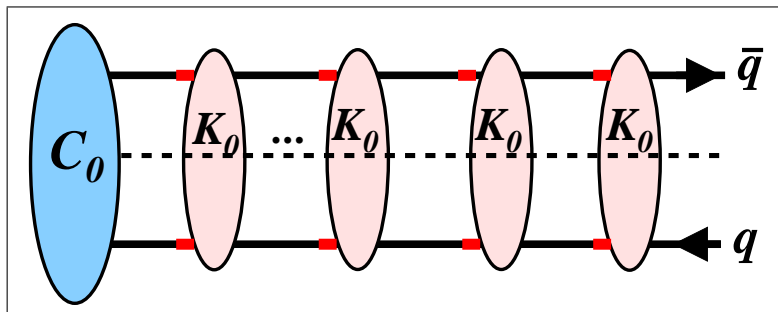
Infinite order analysis if coll. singularities in axial gauge



- $C_0$  is finite,  $K_0, K'_0$  are 2PI kernel. Phase space intact!
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# EGMPR factorization theorem/scheme

## Single ladder



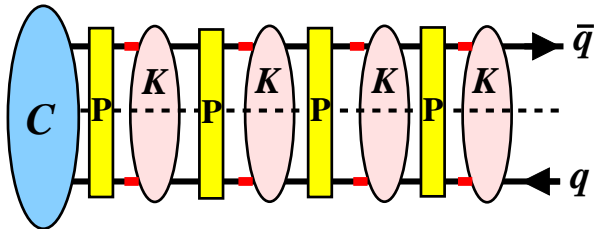
$$F = C_0 \cdot \frac{1}{1 - K_0} = C_0 \cdot (1 + K_0 + K_0 \otimes K_0 + K_0 \otimes K_0 \otimes K_0 + \dots),$$



# Curci-Furmanski-Petronzio (CFP) collinear factorization scheme (1979)

CFP customized EGMPR to  $\overline{MS}$  and exploited it to NLO in practice. Using **casting operator**  $\mathbb{P} = P_{spin} P$  they get:

$$F = C \cdot \frac{1}{1-K} = C_0 \cdot (1 + K + K \otimes K + K \otimes K \otimes K + \dots)$$



where new **finite hard process part** is:

$$C = C_0 \cdot \frac{1}{1-(1-\mathbb{P}) \cdot K_0}$$

The ladder encapsulates all collinear singularities.

**Reorganized kernel** is:

$$K = \mathbb{P} K_0 \cdot \frac{1}{1-(1-\mathbb{P}) \cdot K_0}$$



# Over-subtraction problem in translating CFP/EGMPR into Monte Carlo

From renormalization group eqs. (RGE) or explicit LO calc. we know:

$$\Gamma = e^{+\frac{1}{\varepsilon}} = 1 + \frac{1}{\varepsilon} + \frac{1}{2!} \frac{1}{\varepsilon^2} + \dots$$

Examining up to LO real emissions in CFP scheme we see enormous over-subtractions/cancellations:

$$\Gamma \simeq \frac{1}{1 - \left(1 - e^{-\frac{1}{\varepsilon}}\right)} = 1 + \left(1 - e^{-\frac{1}{\varepsilon}}\right) + \left(1 - e^{-\frac{1}{\varepsilon}}\right)^2 + \dots$$

**NO WAY to build Monte Carlo on that!**

**Need exponent directly from the Feynman diagrams!!!**

Translating  $\varepsilon$ -poles  $\frac{1}{\varepsilon} = \int_0^{\mu_F} \frac{dk^T}{k^T} \left(\frac{k^T}{\mu_F}\right)^\varepsilon$  into ordinary big logs  $\ln \frac{\mu_E}{m_p}$  of EGMPR provides the same picture.



# Correcting for over-subtractions

## New CFT suited for the psMC?

Over-subtraction eliminated thanks to explicit **time ordered exponential** =  $\exp_{TO}$  in the evolution variable = log of the factorization scale (Soper+Nagy at LO):

$$F = \frac{1}{1-K_0} = C_0 \overleftarrow{\mathbb{B}}_\mu \left[ \frac{1}{1-K_0} \right] \cdot \exp_{TO} \left( \overleftarrow{\mathbb{P}}'_\mu \left\{ {}^s K_0 \cdot \overleftarrow{\mathbb{B}}_s \left[ \frac{1}{1-K_0} \right] \right\} \right)$$

$$\overleftarrow{\mathbb{B}}_\mu \left[ \frac{1}{1-K_0} \right] \equiv 1 + \overleftarrow{\mathbb{B}}_\mu [K_0] + \overleftarrow{\mathbb{B}}_\mu [K_0 \cdot K_0] + \overleftarrow{\mathbb{B}}_\mu [K_0 \cdot K_0 \cdot K_0] + \dots$$

Operator  $\overleftarrow{\mathbb{B}}$  is defined **recursively** (similarly as  $\beta$ -functions in Yennie-Frautschi-Suura 1961 subtraction scheme):

$$\overleftarrow{\mathbb{B}}_\mu [K_0] = K_0 - \mathbb{P}'_\mu \{ K_0 \},$$

$$\overleftarrow{\mathbb{B}}_\mu [K_0 \cdot K_0] = K_0 \cdot K_0 - \mathbb{P}'_\mu \{ {}^{s_2} K_0 \} \cdot \mathbb{P}'_{s_2} \{ {}^{s_1} K_0 \} - \mathbb{P}'_\mu \{ {}^{s_2} K_0 \} \cdot \overleftarrow{\mathbb{B}}_{s_2} [K_0] - \overleftarrow{\mathbb{B}}_\mu [K_0] \cdot \mathbb{P}'_\mu \{ K_0 \},$$

$$\overleftarrow{\mathbb{B}}_\mu [K_0 \cdot K_0 \cdot K_0] = K_0 \cdot K_0 \cdot K_0 - \mathbb{P}'_\mu \{ {}^{s_3} K_0 \} \cdot \mathbb{P}'_{s_3} \{ {}^{s_2} K_0 \} \cdot \mathbb{P}'_{s_2} \{ {}^{s_1} K_0 \} - \dots$$

**Modified  $\mathbb{P}'_\mu$  new projection operator is the key point!**

$\mathbb{P} \rightarrow \mathbb{P}'_\mu$  conserves four-momentum, contrary to EGMPR.



# Specs of $\overleftarrow{\mathbb{P}}'_\mu$ modified projection operator

- $\overleftarrow{\mathbb{P}}'_\mu$  does spin projection as  $\mathbb{P}$  of CFP,
- $\overleftarrow{\mathbb{P}}'_\mu(A)$  extracts singular part from integrand  $A$ ,  
(not from the integral  $\int A$  like CFP!)
- where  $A$  is *at most single-log* coll. divergent!
- $\overleftarrow{\mathbb{P}}'_\mu$  acts on integrand, leaves out Lorentz inv. phase space, sets on-shell all (cut) real momenta **towards the hard process**
- $\overleftarrow{\mathbb{P}}'_\mu$  sets upper limit  $\mu$  on the phase space for all real (cut) partons **towards the hadron** using kinematic variable  $s(k_1, \dots, k_n) < \mu$ ,
- examples of  $s(k_1, \dots, k_n)$ : Virtuality, maximum rapidity  $s = \max(k_i^T / \alpha_i)$ , or  $s = \max(k_i^T)$ ,
- Nesting like  $\overleftarrow{\mathbb{P}}'_\mu[K_0 \cdot (1 - \overleftarrow{\mathbb{P}}'(K_0))]$  is allowed.



# Hierarchy of fact. scales in T.O. expon.

Example for up to 3 terms:

$$\begin{aligned} \exp_{TO}(\mathbb{P}'_{\mu}\{A\}) (\mu) = & 1 + \mathbb{P}'_{\mu}\{A\} + \mathbb{P}'_{\mu}\{s_2 A\} \cdot \mathbb{P}'_{s_2}\{s_1 A\} \\ & + \mathbb{P}'_{\mu}\{s_3 A\} \cdot \mathbb{P}'_{s_3}\{s_2 A\} \cdot \mathbb{P}'_{s_2}\{s_1 A\} + \dots \end{aligned}$$

For  $A = \int dLips(k_1, k_2, \dots, k_n) f(k_1, \dots, k_n)$ ,

with  $k_i$  being on-shell emitted partons,

notation  $\{s_3 A\}$  defines  $s_3 = a(a_1, \dots, a_n) = \max(a_1, \dots, a_n)$ .

The entire integrand in

$$\mathbb{P}'_{\mu}\{s_3 A\} \cdot \mathbb{P}'_{s_3}\{s_2 A\} \cdot \mathbb{P}'_{s_2}\{s_1 A\}$$

is multiplied by

$$\theta_{\mu > s_3 > s_2 > s_1}$$

insted of EGMPR/CFP common limit:

$$\theta_{\mu > s_3} \theta_{\mu > s_2} \theta_{\mu > s_1}$$



# Standard inclusive PDFs, evolution eqs. and kernels recovered after Ph.Sp. integration

**Exclusive PDF** (ePDF) is the integrand in:

$$D(\mu) = \exp_{TO} \left( \overleftarrow{\mathbb{P}}'_\mu \left\{ {}^s K_0 \cdot \overleftarrow{\mathbb{B}}_s \left[ \frac{1}{1 - K_0} \right] \right\} \right) = \exp_{TO}(K).$$

LO and NLO truncations of the **exclusive evolution kernel**  $K_\mu$  are:

$$K_\mu^{LO} = \overleftarrow{\mathbb{P}}'_\mu \left\{ {}^s K_0 \right\}, \quad \text{taken at } \mathcal{O}(\alpha^1),$$

$$K_\mu^{NLO} = \overleftarrow{\mathbb{P}}'_\mu \left\{ {}^s K_0 + K_0 \cdot (1 - \overleftarrow{\mathbb{P}}'_s) \cdot K_0 \right\}, \quad \text{truncated at } \mathcal{O}(\alpha^2).$$

**Standard inclusive PDF**  $D(\mu, x)$ , from ph.space integration of ePDF with fixed  $x$ , obeys by construction DGLAP evolution equation:

$$\partial_\mu D(\mu, x) = \mathcal{P} \otimes D(\mu)(x)$$

with the DGLAP **standard inclusive kernel**:

$$\mathcal{P}(x) = \int d\text{Lips} \delta \left( x = \frac{\sum k_i^+}{E_0} \right) \delta \left( 1 - \frac{s}{\mu} \right) \overleftarrow{\mathbb{P}}'_\mu \left\{ {}^s K_0 \cdot \overleftarrow{\mathbb{R}}_s[K_0] \right\}.$$

# Bottom line:

Modified factorization scheme cures both problems of EGMPR/CFP:

- (i) over-subtraction/cancellations and
- (ii) 4-momentum non-conservation.

Ready to go for psMC with complete NLO, both in the hard part and ladder parts.

Important ingredient to be added: kinematic mapping in the modified  $\overleftarrow{\mathbb{P}}'_\mu$ , see next slides on psMC for DY.



**NLO psMC for:**

**$W/Z$  production (Drell-Yan)**

**and for  $ep$  DIS**

**In the following several simplifications are adopted temporarily:**

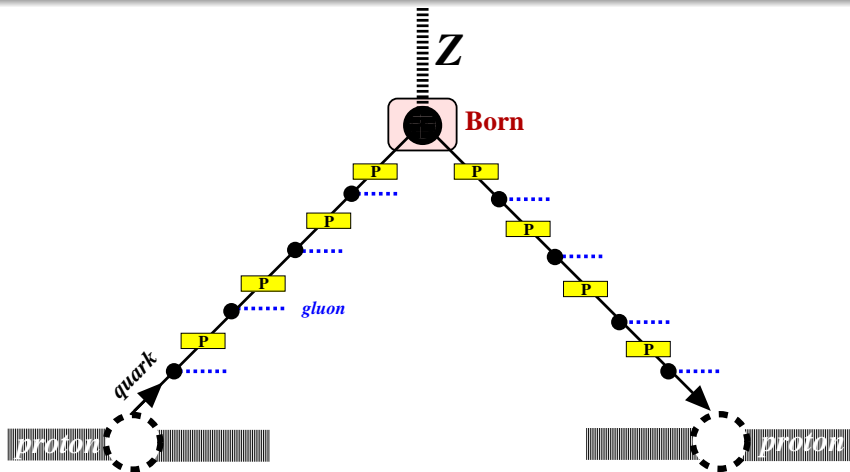
- Non-singlet kernels only,
- Only  $q\bar{q} \rightarrow W/Z$ , omitted  $qg \rightarrow W/Z$
- non-running  $\alpha_S$
- Initial PDFs at low  $\mu = Q$  assumed, but not explicitly shown in the formulae

The notorious gluonstrahlung is our primary target!

**LO psMC is (re-)constructed from the scratch, in a way compatible with our new factorization scheme.**



# LO psMC is (re-)constructed from the scratch



$$\sigma(C_0^{(0)} \Gamma_F^{(1)} \Gamma_B^{(1)}) = \sum_{n_1=1}^{\infty} \sum_{n_2=1}^{\infty} \{ \sigma [ C_0^{(0)} (\mathbb{P}' K_{0F}^{(1)})^{n_1} (\mathbb{P}'' K_{0B}^{(1)})^{n_2} ] \}_{T.O.}$$



# LO psMC for W/Z production: Details

$$\begin{aligned}\sigma(C_0^{(0)}\Gamma_F^{(1)}\Gamma_B^{(1)}) &= \sum_{n_1=0}^{\infty} \sum_{n_2=0}^{\infty} \int d\hat{x}_F d\hat{x}_B d\Xi \\ &\times e^{-S_F} \int_{\Xi > \eta_{n_1}} \left( \prod_{i=1}^{n_1} d^3\mathcal{E}(\bar{k}_i) \theta_{\eta_i > \eta_{i-1}} \frac{2C_F\alpha_s}{\pi^2} \bar{P}(z_{Fi}) \right) \delta_{\hat{x}_F=1-\sum_j \hat{\alpha}_j} \\ &\times e^{-S_B} \int_{\Xi < \eta_{n_2}} \left( \prod_{i=1}^{n_2} d^3\mathcal{E}(\bar{k}_i) \theta_{\eta_i < \eta_{i-1}} \frac{2C_F\alpha_s}{\pi^2} \bar{P}(z_{Bi}) \right) \delta_{\hat{x}_B=1-\sum_j \hat{\beta}_j} \\ &\times \delta\left(\Xi - \frac{\eta_{0F} + \eta_{0B}}{2} - \ln \frac{\hat{x}_F}{\hat{x}_B}\right) d\tau_2(P - \sum_{j=1}^{n_1+n_2} k_j; q_1, q_2) \frac{d\sigma_B}{d\Omega}(s\hat{x}_F\hat{x}_B, \hat{\theta}),\end{aligned}$$

---

$S_F$  and  $S_B$  = Sudakov formfactors,  $\bar{P}(z) = \frac{1}{2}(1+z^2)$ ,  
 $\Xi$  = Rapidity of Z, division plane between F and B hemispheres.  
 $\theta$  = angle of decay products (leptons) in Z rest frame.  
 $\hat{s} = s\hat{x}_F\hat{x}_B$  = effective mass of Z boson.



# LO psMC for W/Z production: Details

$$\begin{aligned}\sigma(C_0^{(0)}\Gamma_F^{(1)}\Gamma_B^{(1)}) &= \sum_{n_1=0}^{\infty} \sum_{n_2=0}^{\infty} \int d\hat{\chi}_F d\hat{\chi}_B d\Xi \\ &\times e^{-S_F} \int_{\Xi > \eta_{n_1}} \left( \prod_{i=1}^{n_1} d^3\mathcal{E}(\bar{k}_i) \theta_{\eta_i > \eta_{i-1}} \frac{2C_F\alpha_s}{\pi^2} \bar{P}(z_{Fi}) \right) \delta_{\hat{\chi}_F=1-\sum_j \hat{\alpha}_j} \\ &\times e^{-S_B} \int_{\Xi < \eta_{n_2}} \left( \prod_{i=1}^{n_2} d^3\mathcal{E}(\bar{k}_i) \theta_{\eta_i < \eta_{i-1}} \frac{2C_F\alpha_s}{\pi^2} \bar{P}(z_{Bi}) \right) \delta_{\hat{\chi}_B=1-\sum_j \hat{\beta}_j} \\ &\times \delta\left(\Xi - \frac{\eta_{0F} + \eta_{0B}}{2} - \ln \frac{\hat{\chi}_F}{\hat{\chi}_B}\right) d\tau_2(P - \sum_{j=1}^{n_1+n_2} k_j; q_1, q_2) \frac{d\sigma_B}{d\Omega}(s\hat{\chi}_F\hat{\chi}_B, \hat{\theta}),\end{aligned}$$

Eikonal phase space for real gluon:

$$d^3\mathcal{E}(k) = \frac{d^3k}{2k^0} \frac{1}{k^2} = \pi \frac{d\phi}{2\pi} \frac{d\alpha}{\alpha} d\eta = \pi \frac{d\phi}{2\pi} \frac{d\beta}{\beta} d\eta,$$

Lightcone variables:  $\alpha = \frac{k^+}{2E}$ ,  $\beta = \frac{k^-}{2E}$ ; rapidity:  $\eta = \frac{1}{2} \ln \frac{k^+}{k^-}$ ,

$d\tau_2(Q; q_1, q_2)$  = two-body phase space element.



# LO psMC for W/Z production: Details

$$\begin{aligned}\sigma(C_0^{(0)}\Gamma_F^{(1)}\Gamma_B^{(1)}) &= \sum_{n_1=0}^{\infty} \sum_{n_2=0}^{\infty} \int d\hat{x}_F d\hat{x}_B d\Xi \\ &\times e^{-S_F} \int_{\Xi > \eta_{n_1}} \left( \prod_{i=1}^{n_1} d^3\mathcal{E}(\bar{k}_i) \theta_{\eta_i > \eta_{i-1}} \frac{2C_F\alpha_s}{\pi^2} \bar{P}(z_{Fi}) \right) \delta_{\hat{x}_F=1-\sum_j \hat{\alpha}_j} \\ &\times e^{-S_B} \int_{\Xi < \eta_{n_2}} \left( \prod_{i=1}^{n_2} d^3\mathcal{E}(\bar{k}_i) \theta_{\eta_i < \eta_{i-1}} \frac{2C_F\alpha_s}{\pi^2} \bar{P}(z_{Bi}) \right) \delta_{\hat{x}_B=1-\sum_j \hat{\beta}_j} \\ &\times \delta\left(\Xi - \frac{\eta_{0F} + \eta_{0B}}{2} - \ln \frac{\hat{x}_F}{\hat{x}_B}\right) d\mathcal{T}_2\left(P - \sum_{j=1}^{n_1+n_2} k_j; q_1, q_2\right) \frac{d\sigma_B}{d\Omega}(s\hat{x}_F\hat{x}_B, \hat{\theta}),\end{aligned}$$

Variables in LO evolution kernels:

$$z_{Fi} = \frac{\hat{x}_{Fi}}{\hat{x}_{F(i-1)}}, \quad \hat{x}_{Fi} = 1 - \sum_{j=1}^i \hat{\alpha}_j = \prod_{j=1}^i z_{Fj},$$

$$z_{Bi} = \frac{\hat{x}_{Bi}}{\hat{x}_{B(i-1)}}, \quad \hat{x}_{Bi} = 1 - \sum_{j=1}^i \hat{\beta}_j = \prod_{j=1}^i z_{Bj},$$

For “mapped” lightcone variables  $\hat{\alpha}_i$  and  $\hat{\beta}_i$ ; see next slide.



# Define hat-variables $\hat{\alpha}_i$ and $\hat{\beta}_i$

Mapping entering definition of  $\mathbb{P}'$  and  $\mathbb{P}'$

Order ALL gluons according to rapidity distance from  $\Xi$ , the position of  $Z$ : Define permutation  $\pi = \{\pi_1, \pi_2, \dots, \pi_{n_1+n_2}\}$  such that  $|\eta_{\pi_i} - \Xi| > |\eta_{\pi_{i-1}} - \Xi|$ ,  $i = 1, \dots, n_1 + n_2$

Define in a *recursive* way dilatation transformation:

$$k_{\pi_i} = \lambda_i \bar{k}_{\pi_i}, \quad \lambda_i = \frac{s(\bar{x}_{i-1} - \bar{x}_i)}{2(P - \sum_{j=1}^{i-1} k_{\pi_j}) \cdot \bar{k}_{\pi_i}}, \quad i = 1, 2, \dots, n_1 + n_2.$$

Rescaling factor  $\lambda_i$  is chosen such that

$$\bar{s}_i = s\bar{x}_i = s \prod_{j=1}^i \hat{Z}_{(F,B)\pi_j} = (P - \sum_{j=1}^i k_{\pi_j})^2 = (P - \sum_{j=1}^i \lambda_j \bar{k}_{\pi_j})^2.$$

In F hemisphere  $\hat{\alpha}_i = \lambda_i \alpha_i$  and in B hemisphere  $\hat{\beta}_i = \lambda_i \beta_i$ .

(An improvement over 1st such scenario publ. in 2007 in APP, Stephes at.al.)



# Overview of the LO Monte Carlo algorithm:

- Variables  $\hat{z}_F$  and  $\hat{z}_B$  are generated by FOAM and  $\Xi$  is determined immediately.
- Four momenta  $\bar{k}_j^\mu$  are generated separately in F and B parts of the phase space using CMC module, with the corresponding constraints  $\sum_{j \in F} \hat{\alpha}_j = 1 - \hat{z}_F$  and  $\sum_{j \in B} \hat{\beta}_j = 1 - \hat{z}_B$ .
- Double ordering permutation  $\pi$  is established.
- Using  $P$  and  $\bar{k}_{\pi_1}$  rescaling parameter  $\lambda_1$  is calculated,  $k_{\pi_1} = \lambda_1 \bar{k}_{\pi_1}$  is set. At this stage  $(P - k_{\pi_1})^2 = s x_1$ , where  $x_1 = z_{\pi_1} = 1 - \hat{\alpha}_{\pi_1}$  or  $x_1 = z_{\pi_1} = 1 - \hat{\beta}_{\pi_1}$ , depending whether  $k_{\pi_1}$  was in F or B part of LIPS.
- Using  $P - k_{\pi_1}$  and  $\bar{k}_{\pi_2}$  parameter  $\lambda_2$  is found and  $k_{\pi_2} = \lambda_2 \bar{k}_{\pi_2}$  is set. At this stage we enforce  $(P - k_{\pi_1} - k_{\pi_2})^2 = s x_2 = s z_{\pi_1} z_{\pi_2}$ . This recursive procedure continues until the last gluon.
- In the rest frame of  $\hat{P} = P - \sum_j k_{\pi_j}$  4-momenta of  $q_1^\mu$  and  $q_2^\mu$  are generated according to Born angular distribution.

Kinematics of the two hemispheres is interrelated very gently, starting from very collinear gluons and finishing with the least collinear ones.



# Exact analytical integration

The above LO MC covers multigluon phase space without any gaps or overlaps.

Moreover, **EXACT analytical integration** of the LO MC distributions over the multigluon phase space is possible:

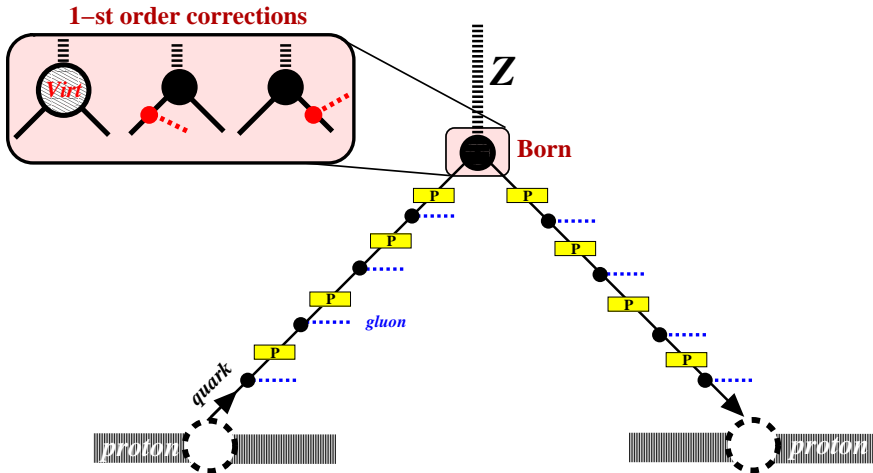
$$\sigma(C_0^{(0)}\Gamma_F^{(1)}\Gamma_B^{(1)}) = \int_0^1 d\hat{x}_F d\hat{x}_B D_F(\Xi, \hat{x}_F) D_B(\Xi, \hat{x}_B) \sigma_B(s\hat{x}_F\hat{x}_B).$$

with two PDFs obeying DGLAP nonsinglet LO evolution equation:

$$\frac{\partial}{\partial \Xi} D_F(\Xi, x) = [\mathcal{P} \otimes D_F(\Xi)](x).$$



# psMC with LO ladder and NLO hard process



# NLO Monte Carlo weight

This is Yennie-Frautschi-Suura (YFS) style!

Once LO MC is re-designed, introduction of the complete NLO to hard process part is done with help of **simple positive MC weight**:

$$W_{MC}^{NLO} = 1 + \Delta_{S+V} + \sum_{j \in F} \frac{\tilde{\beta}_1(\hat{s}, \hat{p}_F, \hat{p}_B; a_j, z_{Fj})}{\bar{P}(z_{Fj}) d\sigma_B(\hat{s}, \hat{\theta})/d\Omega} + \sum_{j \in B} \frac{\tilde{\beta}_1(\hat{s}, \hat{p}_F, \hat{p}_B; a_j, z_{Bj})}{\bar{P}(z_{Bj}) d\sigma_B(\hat{s}, \hat{\theta})/d\Omega},$$

where the IR/Col.-finite **real** emission part is

$$\tilde{\beta}_1(\hat{p}_F, \hat{p}_B; q_1, q_2, k) = \left[ \frac{(1-\alpha)^2}{2} \frac{d\sigma_B}{d\Omega_q}(\hat{s}, \theta_{F1}) + \frac{(1-\beta)^2}{2} \frac{d\sigma_B}{d\Omega_q}(\hat{s}, \theta_{B2}) \right] \\ - \theta_{\alpha>\beta} \frac{1 + (1-\alpha-\beta)^2}{2} \frac{d\sigma_B}{d\Omega_q}(\hat{s}, \hat{\theta}) - \theta_{\alpha<\beta} \frac{1 + (1-\alpha-\beta)^2}{2} \frac{d\sigma_B}{d\Omega_q}(\hat{s}, \hat{\theta}),$$

and the kinematics independent **virtual+soft** correction is

$$\Delta_{V+S} = \frac{C_F \alpha_s}{\pi} \left( \frac{1}{3} \pi^2 - 4 \right) + \frac{C_F \alpha_s}{\pi} \frac{1}{2}$$

Next slide more on  $\Delta_{V+S}$ .



# More on $\Delta_{V+S}$ virtual+soft correction

$$\Delta_{V+S} = D_{DY}^{\overline{MS}}(z) - 2C_{ct}^{psMC}(z)$$

where we use  $\overline{MS}$  results, eq. (89) in Altarelli+Ellis+Martinelli (1979):

$$D_{DY}^{\overline{MS}}(z) = \delta(1-z) + \delta(1-z) \frac{C_F \alpha_s}{\pi} \left( \frac{1}{3} \pi^2 - 4 \right) + 2 \frac{C_F \alpha_s}{\pi} \left( \frac{\hat{s}}{\mu^2} \right)^\epsilon \left( \frac{\bar{P}(z)}{1-z} \right)_+ \left( \frac{1}{\epsilon} + \gamma_E - \ln 4\pi + [2 \ln(1-z) - \ln z] \right)$$

and collinear counterterm of psMC (one gluon in psMC in  $d = 4 + 2\epsilon$ ):

$$C_{ct}^{psMC}(z) = \frac{C_F \alpha_s}{\pi} \int_{\beta < \alpha} \frac{d\alpha d\beta}{\alpha\beta} \int d\Omega_{1+2\epsilon} \left( \frac{s\alpha\beta}{\mu_F^2} \right)^\epsilon \bar{P}(1-\alpha, \epsilon) \delta_{1-z=\alpha} = \frac{C_F \alpha_s}{\pi} \left( \frac{\bar{P}'(z, \epsilon)}{1-z} \right)_+ \left( \frac{1}{\epsilon} + \gamma_E - \ln 4\pi + \ln \frac{s}{\mu_F^2} \right),$$

$$\bar{P}'(z, \epsilon) = \bar{P}(z) + \frac{1}{2} \epsilon (1-z)^2 + \epsilon \ln(1-z).$$



# Exact analytical integration at NLO

**EXACT** analytical integration of the NLO MC distributions over the multigluon phase space is again possible(!):

$$\sigma(C_0^{(1)}\Gamma_F\Gamma_B) = \int_0^1 d\hat{x}_F d\hat{x}_B dz D_F(\Xi, \hat{x}_F) D_B(\Xi, \hat{x}_B) \sigma_B(SZ\hat{x}_F\hat{x}_B) \\ \times \left\{ \delta_{z=1}(1 + \Delta_{S+V}) + C_{2r}^{psMC}(z) \right\}$$

where

$$C_{2r}^{psMC}(z) = \frac{2C_F\alpha_s}{\pi} \left[ -\frac{1}{2}(1-z) \right]$$

The above differs from  $\overline{MS}$  eq. (90) in Altarelli-Ellis-Martinelli (79)

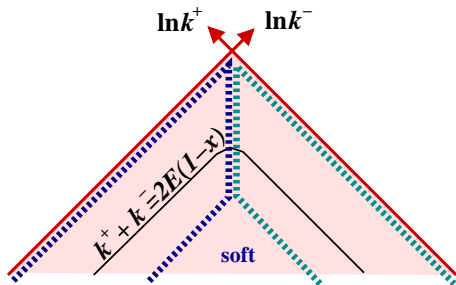
$$C_{2r}^{\overline{MS}}(z) = 2\frac{C_F\alpha_s}{\pi} \left( \frac{\bar{P}(x)}{1-z} \right) [2\ln(1-z) - \ln z]$$

Why? psMC factorization scheme is slightly different from  $\overline{MS}$ :

$$C_{2r}^{psMC}(z) - C_{2r}^{\overline{MS}}(z) = -2C_{ct}^{psMC}(z) + 2C_{ct}^{\overline{MS}}(z) \simeq 4\frac{C_F\alpha_s}{\pi} \left( \frac{\bar{P}(x)}{1-z} \ln(1-x) \right)$$

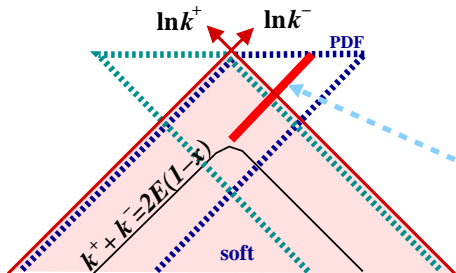
# Difference between $\overline{MS}$ and psMC fact. schemes

Simple kinematics explains  $4 \ln(1-x)/(1-x)_+$



psMC fact. scheme:

$$0 \frac{|\ln(1-x)|}{1-x}$$



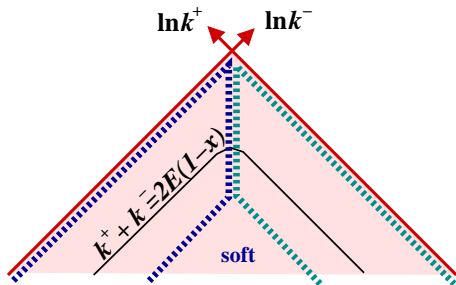
$\overline{MS}$  fact. scheme:

$$\int_{1-x}^{1/(1-x)} \frac{d\beta}{\beta} = 2 \frac{|\ln(1-x)|}{1-x}$$



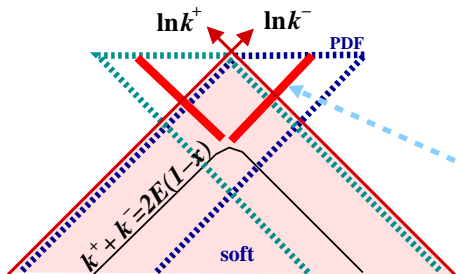
# Difference between $\overline{MS}$ and psMC fact. schemes

Simple kinematics explains  $4 \ln(1-x)/(1-x)_+$



psMC fact. scheme:

$$0 \frac{|\ln(1-x)|}{1-x}$$



$\overline{MS}$  fact. scheme:

$$\int_{1-x}^{1/(1-x)} \frac{d\beta}{\beta} = 4 \frac{|\ln(1-x)|}{1-x}$$



# Summary on differences with MC@NLO

- Very simple and positive MC weight adding NLO on top of LO psMC
- No need to correct for the difference in collinear counterterm of psMC and  $\overline{MS}$  schemes.
- Virtual+soft corrections is completely kinematics independent
  - all their annoying  $d\Sigma^{c\pm}$  contributions are simply gone!
- Built in resummation of  $\frac{\ln^n(1-x)}{1-x}$  terms.
- more transparent relation to CFTs

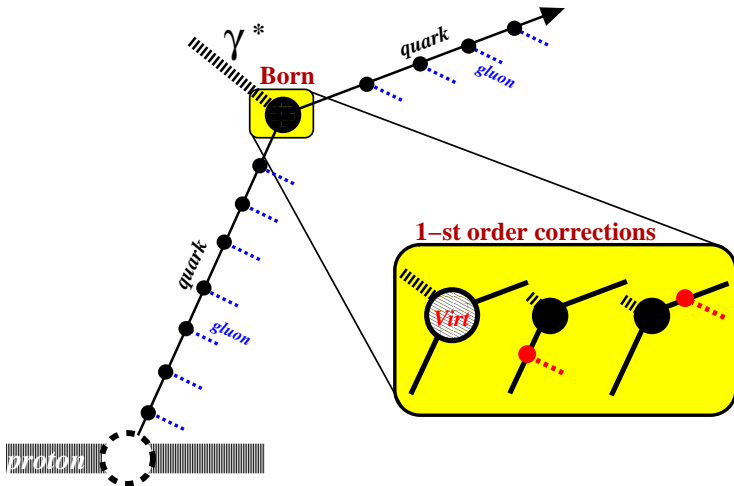
## All the above goodies at the price of:

- rebuilding LO psMC from the scratch
- and slight (but well controlled) change of the factorization scheme in the subtracted hard process ME and in PDFs



# NLO corrections to hard process in $ep$ DIS

<http://jadach.web.cern.ch/jadach/public/CERNjuly2010.pdf>  
describes LO+NLO psMC for DIS process in some detail



Below we only comment on factorization scheme dependence.



# NLO corrections to hard process in $ep$ DIS

Here in the NLO psMC weight for hard process:

$$\Delta_{V+S} = D_{DIS}^{\overline{MS}}(z) - 2C_{ct}^{psMC}(z) \equiv 0,$$

(exploiting again  $\overline{MS}$  calculation of Altarelli+Ellis+Martinelli (1979)), while analytical integration of the multigluon LO+NLO psMC distributions yields the following structure functions:

$$\frac{d\sigma_{DIS}^{NLO}}{dt dx_B} = \frac{2\pi\alpha^2 Q_q^2}{t^2} \int dx dz \delta_{x_B=xz} D_I(t, x) \left[ W_0(y_B) C_2^{psMC}(z) + 2(1-y_B)z^{-1} C_L(z) \right],$$

$$C_2^{psMC}(z) = \frac{C_F\alpha_s}{\pi} \left\{ -\frac{1+z^2}{2(1-z)} \ln(1-z) - \frac{3}{4} \frac{1}{1-z} + \frac{3}{2} + z \right\}_+$$

$$C_L(z) = \frac{C_F\alpha_s}{\pi} z^2, \quad W_0(y) = 1 + (1-y)^2, \quad y_B = t/(sx_B),$$

and  $x_B$  is standard Bjorken variable.

The difference with the standard  $C_2$  of  $\overline{MS}$  (Bardin+Buras, 78) is entirely due to difference between collinear counterterms of psMC and  $\overline{MS}$ .

NB Altarelli-Ellis-Martinelli FS-independent difference DY-2\*DIS is reproduced by two above psMCs up to NLO exactly!

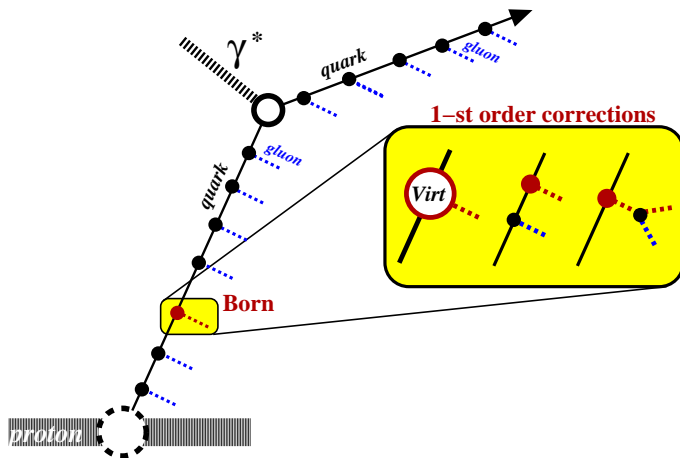
## NLO corrections to the ladder parts of psMC



# Leading Order (LO) ladder vertex is our “Born”

Emission of gluons out of quark

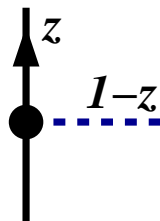
Again starting from Feynman diagrams of Curci-Furmanski-Petronzio scheme (axial gauge), and recalculating their NLO DGLAP kernels.



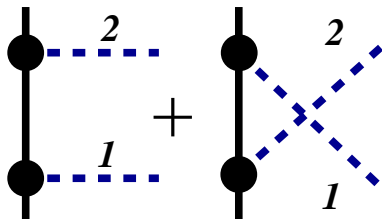
# 1-st order virtual and real correction (subset) diagrams

Virtual :

$$\left(1 + \Delta_{ISR}^{(1)}(z)\right)$$



Real :



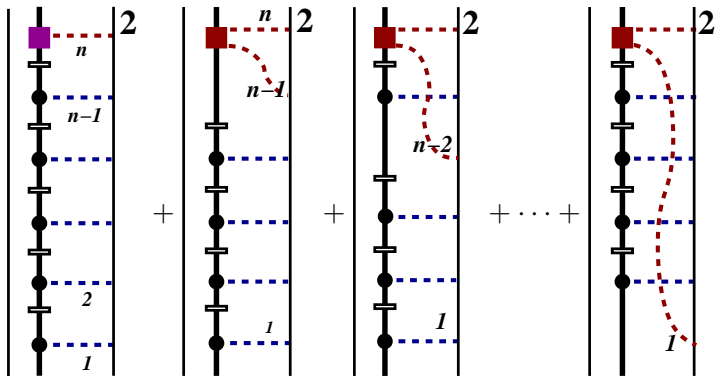


# LO ladder = parton shower MC

$$\sum_{n=0}^{\infty} = e^{-S_{ISR}} \sum_{n=0}^{\infty} \prod_{i=1}^n \frac{d^3 k_i}{k_i^0} \theta_{Q > a_i > a_{i-1}} \rho_{1B}^{(0)}(k_i) \delta_{x = \prod z_i}$$

$$a_i = \frac{k_i^T}{\alpha_i}, \quad \alpha_i = \frac{k_i^+}{2E_h}, \quad \rho_{1B}^{(0)}(k_i) = \frac{2C_F^2 \alpha_s}{\pi} \frac{1}{k_i^{T2}} \frac{1+z^2}{2}$$

# LO with NLO-corrected kernel at the end of the ladder



Virt. multiplicative

Undoing LO simplificat.

Sum over trailing LO spectators, essential (BE, YFS61)

$$\left| \begin{array}{c} \uparrow \\ \text{---} \\ \square \\ \text{---} \\ \uparrow \end{array} \right|^2 = (1 + 2\Re(\Delta_{ISR}^{(1)})) \left| \begin{array}{c} \uparrow \\ \text{---} \\ \bullet \\ \text{---} \\ \uparrow \end{array} \right|^2,$$

$$\left| \begin{array}{c} \uparrow \\ \text{---} \\ \square \\ \text{---} \\ \uparrow \end{array} \right|^2 = \left| \begin{array}{c} \bullet \\ \text{---} \\ \bullet \\ \text{---} \\ \bullet \\ \text{---} \\ \bullet \end{array} \right|^2 + \left| \begin{array}{c} \bullet \\ \text{---} \\ \bullet \\ \text{---} \\ \bullet \\ \text{---} \\ \bullet \end{array} \right|^2 - \left| \begin{array}{c} \bullet \\ \text{---} \\ \square \\ \text{---} \\ \bullet \end{array} \right|^2$$

# LO with NLO-corrected end-ladder kernel, $\sim C_F^2$

## MORE DETAILS:

$$\bar{D}_B^{[1]}(x, Q) = e^{-S_{ISR}} \left[ \text{Diagram 1} + e^{-S_{ISR}} \sum_{j=1}^{n-1} \text{Diagram 2} \right] = e^{-S_{ISR}} \left\{ \delta_{x=1} + \right.$$

$$\left. + \sum_{n=1}^{\infty} \left( \prod_{i=1}^n \int_{Q > a_i > a_{i-1}} d^3 \eta_i \rho_{1B}^{(1)}(k_i) \right) \left[ \beta_0^{(1)}(z_n) + \sum_{j=1}^{n-1} W(\tilde{k}_n, \tilde{k}_j) \right] \delta_{x=\prod_{j=1}^n x_j} \right\},$$

where  $d\eta_i = \frac{d^3 k_i}{k_i^0}$ ,  $\beta_0^{(1)} = \frac{\text{Diagram 3}}{\text{Diagram 4}}$ ,  $W(k_2, k_1) = \frac{\text{Diagram 5} + \text{Diagram 6}}{\text{Diagram 4}} - 1$ .

Mapping  $k_i \rightarrow \tilde{k}_i$  instrumental.  $S_{ISR}$  = double-log Sudakov,  $W$  is non-singular!



# Algebraic crosscheck

Analytical integration of NLO part  $\sum_j W(\tilde{k}_n, \tilde{k}_j)$  can be done leading to:

$$\sum_{n=1}^{\infty} \int du \int_{Q > a_n > a_{n-1}} \frac{da_n}{a_n} \mathcal{P}_{qq}^{(1)}(u) \left( \prod_{i=1}^{n-1} \int_{a_{i+1} > a_i > a_{i-1}} \frac{da_i}{a_i} \mathcal{P}_{qq}^{(0)}(z_i) \right) \delta_{x=u \prod_{j=1}^{n-1} z_j}.$$

where we recover precisely NLO part (including virtuals) of standard DGLAP kernel  $\mathcal{P}_{qq}^{(1)}(u)$  defined according to:

$$\mathcal{P}_{qq}^{(1)}(u) \ln \frac{Q}{q_0} = \int_{Q > a_n > a_0} d^3 \eta_n \rho_{1B}^{(1)}(k_n) \beta_0^{(1)}(z_n) \delta_{u=z_n} + \int_{Q > a_n > a_0} d^3 \eta_n \int_{a_n > a_{n'} > 0} d^3 \eta_{n'} \beta_1^{(1)}(\tilde{k}_n, \tilde{k}_{n'}) \delta_{u=z_n z_{n'}}.$$

One NLO standard inclusive kernel of DGLAP truly reproduced, modulo factorization scheme change  $\overline{MS} \rightarrow \text{psMC}$ .



# NLO-corrected middle-of-the-ladder kernel, $\sim C_F^2$

Position of the NLO correction/insertion  $p$  can be anywhere in the ladder and we sum up over  $p$ :

$$\bar{D}_B^{[1]}(x, Q) = e^{-S_{ISR}} \sum_{n=0}^{\infty} \left\{ \sum_{p=1}^n \left[ \text{Diagram 1} \right] + \sum_{p=1}^n \sum_{j=1}^{p-1} \left[ \text{Diagram 2} \right] \right\} = e^{-S_{ISR}} \left\{ \delta_{x=1} + \right.$$

$$\left. + \sum_{n=1}^{\infty} \left( \prod_{i=1}^n \int_{Q > a_i > a_{i-1}} d^3 \eta_i \rho_{1B}^{(1)}(k_i) \right) \left[ \sum_{p=1}^n \beta_0^{(1)}(z_p) + \sum_{p=1}^n \sum_{j=1}^{p-1} W(\tilde{k}_p, \tilde{k}_j) \right] \delta_{x=\prod_{j=1}^n x_j} \right\},$$

Next step is to add more “NLO insertions”,  
 for instance 2 at positions  $p_1$  and  $p_2$  and sum up over them...  
 then 3 insertions at  $p_1, p_2, p_3$  and so on  
 – LO+NLO kernels built up all over along the ladder!



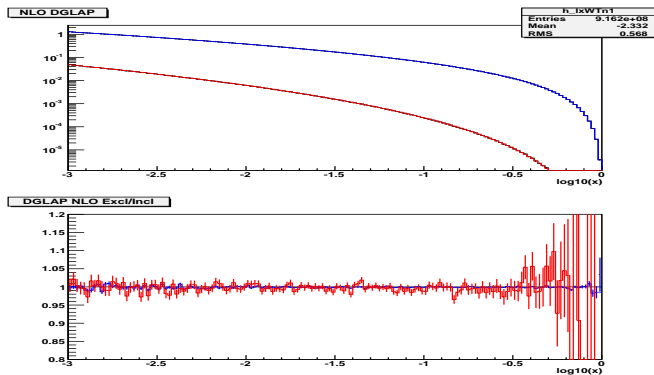
# NLO-corrected kernels all over the ladder, $\sim C_F^2$

$$\begin{aligned}
 \bar{D}_B^{[1]}(x, Q) &= e^{-S_{ISR}} \sum_{n=0}^{\infty} \left\{ \begin{array}{l} \text{Diagram 1: Ladder with } n \text{ rungs, rungs } p, 2, 1 \text{ highlighted.} \\ \text{Diagram 2: Ladder with } n \text{ rungs, rungs } p_1, j_1 \text{ highlighted.} \\ \text{Diagram 3: Ladder with } n \text{ rungs, rungs } p_1, p_2, j_1, j_2 \text{ highlighted.} \end{array} \right\} \\
 &= e^{-S_{ISR}} \left\{ \delta_{x=1} + \sum_{n=1}^{\infty} \left( \prod_{i=1}^n \int_{Q > a_i > a_{i-1}} d^3 \eta_i \rho_{1B}^{(1)}(k_i) \beta_0^{(1)}(z_p) \right) \left[ 1 + \sum_{p=1}^n \sum_{j=1}^{p-1} W(\tilde{k}_p, \tilde{k}_j) + \right. \right. \\
 &\quad \left. \left. + \sum_{p_1=1}^n \sum_{p_2=1}^{p_1-1} \sum_{\substack{j_1=1 \\ j_1 \neq p_2}}^{p_1-1} \sum_{\substack{j_2=1 \\ j_2 \neq p_1, j_2}}^{p_2-1} W(\tilde{k}_{p_1}, \tilde{k}_{j_1}) W(\tilde{k}_{p_2}, \tilde{k}_{j_2}) + \dots \right] \delta_{x=\prod_{j=1}^n x_j} \right\},
 \end{aligned}$$

The above has been tested with 3-digit precision in the MC prototype, see next slide.



# Numerical test of ISR pure $C_F^2$ NLO MC



Numerical results for  $D(x, Q)$  from inclusive and exclusive **two** Monte Carlos. **Blue curve** is single NLO insertion, **red curve** is double insertion component. LO+NLO is off scale. Evolution  $10\text{GeV} \rightarrow 1\text{TeV}$  starting from  $\delta(1-x)$ . The ratio demonstrates 3-digit agreement, in units of LO.



# THE PROBLEM WITH GLUON PAIR COMPONENT OF the NLO KERNEL, $\sim C_F C_A$ (FSR)

Straightforward inclusion of gluon pair diagram in the previous method would ruin Monte Carlo weight due to presence of Sudakov double logarithmic  $+S_{FSR}$  in 2-real correction:

$$\left| \text{diagram with red square} \right|^2 = \left| \text{diagram 1} + \text{diagram 2} + \text{diagram 3} \right|^2 - \left| \text{diagram with white box} \right|^2$$

and  $-S_{FSR}$  in the virtual correction:

$$\left| \text{diagram with purple square} \right|^2 = (1 + 2\Re(\Delta_{ISR} + V_{FSR})) \left| \text{diagram with black dot} \right|^2.$$

SOLUTION: Resummation/exponentiation of FSR, see next slides for details of the scheme and numerical test of the prototype MC.



# NLO FSR corr. at the end of the ladder, $\sim C_F C_A$

Additional NLO FSR corr. at the end of the ladder:

$$e^{-S_{ISR} - S_{FSR}} \sum_{n,m=0}^{\infty} \sum_{r=1}^m \left| \begin{array}{c} \text{Diagram with } n \text{ rungs and } m \text{ vertices} \\ \text{with } r \text{ real gluons} \end{array} \right|^2$$

where Sudakov  $S_{FSR}$  is subtracted in the virtual part:

$$\left| \begin{array}{c} \text{Virtual diagram} \end{array} \right|^2 = (1 + 2\Re(\Delta_{ISR} + V_{FSR} - S_{FSR})) \left| \begin{array}{c} \text{Virtual diagram with } z \text{ and } 1-z \end{array} \right|^2$$

and FSR counterterm is subtracted in the 2-real-gluon part:

$$\left| \begin{array}{c} \text{2-real-gluon diagram} \end{array} \right|^2 = \left| \begin{array}{c} \text{Diagram 1} \end{array} \right|^2 + \left| \begin{array}{c} \text{Diagram 2} \end{array} \right|^2 + \left| \begin{array}{c} \text{Diagram 3} \end{array} \right|^2 - \left| \begin{array}{c} \text{Diagram 4} \end{array} \right|^2 - \left| \begin{array}{c} \text{Diagram 5} \end{array} \right|^2$$

The miracle: both are free of any collinear or soft divergence!!!



# ISR+FSR NLO scheme, NLO corr. at end of the ladder

$$\bar{D}_{NS}^{[1]}(x, Q) =$$

$$e^{-S} \sum_{n,m=0}^{\infty} \left\{ \left| \begin{array}{c} \text{Diagram 1: } n \text{ vertices, } m \text{ external lines} \\ \text{Diagram 2: } n-1 \text{ vertices, } m \text{ external lines} \\ \text{Diagram 3: } n-2 \text{ vertices, } m \text{ external lines} \\ \vdots \\ \text{Diagram } j: \text{ } j \text{ vertices, } m \text{ external lines} \\ \vdots \\ \text{Diagram } m: \text{ } m \text{ vertices, } m \text{ external lines} \end{array} \right. \right\}$$

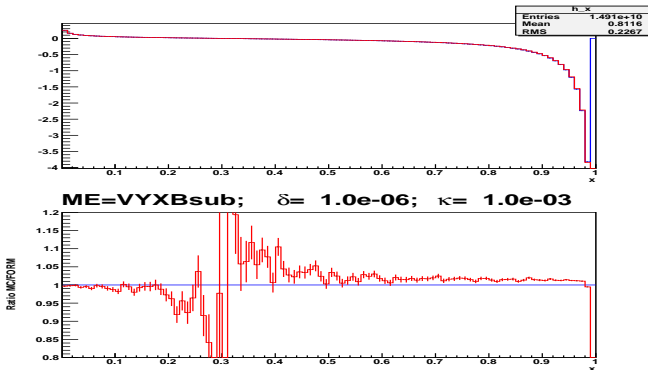
$$= e^{-S_{ISR}} \left\{ \delta_{x=1} + \sum_{n=1}^{\infty} \left( \prod_{i=1}^n \int_{Q > a_i > a_{i-1}} d^3 \eta_i \rho_{1B}^{(1)}(k_i) \right) e^{-S_{FSR}} \sum_{m=0}^{\infty} \left( \prod_{j=1}^m \int_{Q > a_{nj} > a_{n(l-1)}} d^3 \eta'_j \rho_{1V}^{(1)}(k'_j) \right) \right. \\ \left. \times \left[ \beta_0^{(1)}(z_n) + \sum_{j=1}^{n-1} W(\tilde{k}_n, \tilde{k}_j) + \sum_{r=1}^m W(\tilde{k}_n, \tilde{k}'_r) \right] \delta_{x=\prod_{j=1}^n x_j} \right\}$$

$$\beta_0^{(1)} \equiv \frac{\left| \begin{array}{c} \text{Diagram 1: } n \text{ vertices, } m \text{ external lines} \\ \text{Diagram 2: } n-1 \text{ vertices, } m \text{ external lines} \end{array} \right|^2}{\left| \begin{array}{c} \text{Diagram 3: } n \text{ vertices, } m \text{ external lines} \end{array} \right|^2}, \quad W(k_2, k_1) \equiv \frac{\left| \begin{array}{c} \text{Diagram 1: } n \text{ vertices, } m \text{ external lines} \\ \text{Diagram 2: } n-1 \text{ vertices, } m \text{ external lines} \end{array} \right|^2}{\left| \begin{array}{c} \text{Diagram 3: } n \text{ vertices, } m \text{ external lines} \end{array} \right|^2} - 1.$$



# 3-digit precision numerical test of FSR methodology

Numerical test done for single NLO ISR+FSR insertion  
for  $n = 1, 2$  ISR gluons and infinite no. of FSR gluons:



because in this case analytical integration is feasible.  
MC agrees precisely with the analytical result.



# Summary and Prospects

- New formulation of the collinear factorization, better suited for Monte Carlo implementation is defined.
- NLO contributions to hard process and evolution kernels are already recalculated up to NLO in the new scheme (non-singlet NLO exclusive kernels calculated).
- Implementation in the Monte Carlo is tested at the prototype level; critical MC weights are examined numerically.
- R&D phase (almost) completed, MC realization for W/Z prod. at LHC/Tevatron and DIS proc. becomes main front.

