Coherent Exclusive EXponentiation CEEX:
The Case of the Resonant e^+e^- Collision

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- Exclusive exponentiation (EEX) in Monte Carlo, YFS-style, is very successful but limitations:

    1. QED interferences, for example ISR*FSR,
    2. Spin polarization, especially transverse,
    3. Exact matrix element for 2, 3 large p_T photons (background for searches)

    difficult to implement.

- The ultimate solution: Exponentiation re-formulated in terms of spin amplitudes!

Papers:

S. Jadach, B.F.L. Ward and Z. Wąs, CERN-TH/98-235

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Introduction

Deficiencies of YFS **Exclusive Exponentiation** technique, as implemented in KORALZ/YFS3, BHLUMI, BHWIDE, KORALW:

- Lack of the ISR/FSR or Up/Down interferences (except BHWIDE, but difficult to upgrade...)
- Simplified treatment of spin, neglected transverse polarization (except KORALB but exponentiation...)
- Approximate matrix element for 2 and 3 hard large p_T photons. (Important for searches.)

The CEEX solution

- ISR/FSR interferences are included in a natural way. Spin amplitudes for ISR and FSR are summed and squared numerically (the BHWIDE approach gets cumbersome beyond first order)
- Complete exact treatment of fermion spin polarizations (transverse and longitudinal) at the density matrix level (numerically) including Wigner rotations (necessary for interfacing with decay M.C. simulating fermion decays).
- Exact matrix element for 2 and 3 and more photons using Kleiss-Stirling (KS) spinor technique. The entire matrix element for n photons is calculated using KS method.

Kleiss-Stirling spinors

All MASSLESS spinors are transformed out of two *basic constant* spinors $\mathfrak{u}_\pm(\zeta)$, $\zeta^2 = 0$:

$u_\lambda(p) = \frac{1}{\sqrt{2p\cdot\zeta}}\, \not{p}\,\mathfrak{u}_{-\lambda}(\zeta)$, **where** $\mathfrak{u}_+(\zeta) = \not{\eta}\,\mathfrak{u}_-(\zeta),\ \eta^2 = -1,\ (\eta\zeta) = 0.$

The usual relations hold:

$$\not{\zeta}\,\mathfrak{u}_\lambda(\zeta) = 0, \omega_\lambda \mathfrak{u}_\lambda(\zeta) = \mathfrak{u}_\lambda(\zeta), \mathfrak{u}_\lambda(\zeta)\bar{\mathfrak{u}}_\lambda(\zeta) = \not{\zeta}\,\omega_\lambda,$$

$\not{p}\,u_\lambda(p) = 0, \omega_\lambda u_\lambda(p) = u_\lambda(p), u_\lambda(p)\bar{u}_\lambda(p) = \not{p}\,\omega_\lambda$, **where** $\omega_\lambda = \frac{1}{2}(1 + \lambda\gamma_5)$.

Spinors for the MASSIVE particle with four momentum p ($p^2 = m^2$) and spin projection $\lambda/2$ are defined in terms of massless spinors:

$u(p,\lambda) = u_\lambda(p_\zeta) + \frac{m}{\sqrt{2p\zeta}}\mathfrak{u}_{-\lambda}(\zeta)$ **and** $v(p,\lambda) = u_{-\lambda}(p_\zeta) - \frac{m}{\sqrt{2p\zeta}}\mathfrak{u}_\lambda(\zeta),$

where $p_{i\zeta} \equiv \hat{p}_i \equiv p_i - \zeta\, m_i^2/(2\zeta p_i)$.

We shall often exploit the completeness relations:

$$\not{p} + m = \sum_\lambda u(p,\lambda)\bar{u}(p,\lambda), \qquad \not{p} - m = \sum_\lambda v(p,\lambda)\bar{v}(p,\lambda)$$

Toolbox for Kleiss-Stirling spinors

The *inner product* of the two *massless* spinors is defined as follows:

$$s_+(p_1,p_2) \equiv \bar{u}_+(p_1)u_-(p_2), \quad s_-(p_1,p_2) \equiv \bar{u}_-(p_1)u_+(p_2) = -(s_+(p_1,p_2))^*.$$

In any reference frame it can be evaluated using the Kleiss-Stirling expression:
$s_+(p,q) = 2\ (2p\zeta)^{-1/2}\ (2q\zeta)^{-1/2}\ [(p\zeta)(q\eta) - (p\eta)(q\zeta) - i\epsilon_{\mu\nu\rho\sigma}\zeta^\mu\eta^\nu p^\rho q^\sigma]$

For example, if in LAB frame $\zeta = (1,1,0,0)$ and $\eta = (0,0,1,0)$, then in this frame:
$s_+(p,q) = -(q^2 + iq^3)\sqrt{(p^0-p^1)/(q^0-q^1)} + (p^2+ip^3)\sqrt{(q^0-q^1)/(p^0-p^1)}.$

The *inner product* of the two *massive* spinors is:

$$\bar{u}(p_1,\lambda_1)u(p_2,\lambda_2) = S(p_1,m_1,\lambda_1,p_2,m_2,\lambda_2),$$
$$\bar{u}(p_1,\lambda_1)v(p_2,\lambda_2) = S(p_1,m_1,\lambda_1,p_2,-m_2,-\lambda_2),$$
$$\bar{v}(p_1,\lambda_1)u(p_2,\lambda_2) = S(p_1,-m_1,-\lambda_1,p_2,m_2,\lambda_2),$$
$$\bar{v}(p_1,\lambda_1)v(p_2,\lambda_2) = S(p_1,-m_1,-\lambda_1,p_2,-m_2,-\lambda_2),$$

where

$$S(p_1,m_1,\lambda_1,p_2,m_2,\lambda_2) = \delta_{\lambda_1,-\lambda_2} s_{\lambda_1}(p_{1\zeta},p_{2\zeta}) + \delta_{\lambda_1,\lambda_2}\left(m_1\sqrt{\frac{2\zeta p_2}{2\zeta p_1}} + m_2\sqrt{\frac{2\zeta p_1}{2\zeta p_2}}\right)$$

and $p_{i\zeta} \equiv \hat{p_i} \equiv p_i - \zeta\ m_i^2/(2\zeta p_i)$

Why do we need GPS?

GOODIES of the Wigner-Jacob-Wick spinology we know and love:
Elegant definition of spin state and scattering matrix element

$$|p,n>=T(L(p))|\overset{\circ}{p},n>,\quad \mathcal{M}_{nm}=<f|S|i>=<q,m|S|p,n>$$

Simple transformation under rotation using Wigner $\mathcal{D}^s$-matrices

$$T(R)\;|\overset{\circ}{p},n>=\sum_{n'}|\overset{\circ}{p},n'>\mathcal{D}^s_{n'n}(R)$$

Well defined (so called) Wigner-Wick rotation

$$\mathcal{M}_{nm}=\sum_{m',n'}\mathcal{M}_{n'm'}\mathcal{D}^s_{n'n}(R^W_i)\mathcal{D}^{s\,+}_{m'm}(R^W_f)$$

where R^{W} is rotation reflecting directly the change of the quantization axes (in rest frame of the massive particle) or is "generated" by the general Lorentz transformation, exactly as in the paper of Wick (Ann. Phys. **18** (1962) 65) for the Jacob-Wick helicity states.

How to preserve the above nice features within familiar spinor technique? For KS spinors the answer is in our GPS rules.

WHY BOTHER? For unstable fermions like τ production and decay spin amplitudes are calculated using completely different methods and with different quantization axes. We need to know very precisely the spin quantization axes for KS spinors in the M.C. event generation, especially for unstable fermions.

GPS – preliminaria

The necessary condition for validity of the usual Wigner-Jacob-Wick technology (including standard Clebsch-Gordan coefficients etc.) is the textbook phase relation

$$(J_x \pm iJ_y)|m >= [(s \mp m)(s \pm m + 1)]^{1/2}| \pm m >$$

(National Bureau of Standards 1951)

equivalent to a condition with the prosaic rotation by $+\pi$ around y-axis

$\exp(-\frac{1}{2}\, i\pi\, J_y)|+ >= |- >,$ in our $s = 1/2$ case.

Take Weyl representation. Let us define Primary Reference Frame PRF where $\zeta = (1, 0, 0, -1)$ and $\eta = (0, 1, 0, 0)$, (in general PRF≠LAB). If the massive fermion is at rest in the PRF then for KS spinors $u(p, \pm)$ and $v(p, \pm)$ the above phase relation is fulfilled, i.e., spin is quantized using z-axis; the x and y axes are correctly positioned!

In the general case of the fermion in flight the GPS rule will tell us where the quantization z-axis is, and also the x and y axes relevant for the relative phases of $|\pm >$. See next slide.

GPS stands for Global Positioning of Spin

The rules for determining all three spin quantization axes for $u(p, \pm)$ and $v(p, \pm)$ defined with KS method for Weyl representation are the following:

- In the rest frame of the fermion, take the z-axis along $-\vec{\zeta}$.
- Place the x-axis in the plane defined by the z-axis from the previous point and the vector $\vec{\eta}$, in the same half-plane as $\vec{\eta}$.
- With the y-axis, complete the right-handed system of coordinates.

We call the above rules the GPS rules, and we shall call the Spin Quantization Reference Frame (SQRF) determined by the above GPS rules the GPS frame of the fermion.

The formal proofs of the above rules is in CERN-TH-98-235. It amounts to showing that in GPS fermion rest frame the two *constant basic* spinors $\mathfrak{u}_{\pm}(\zeta)$, up to a real constant, are the same (have the same components) as in PRF.

GPS at work: polarized τ decay

Suppose, that for $\tau(p) \to X$ decay, using Feynman rules and classic "Dirac alchemy", including $u(p,s)\,\bar{u}(p,s) = \frac{1}{2}(1+\gamma_5 \not{s})\,(\not{p}+m)$, we obtain (def. polarimeter vector h):

$$d\Gamma^{\rm pol.}_{class.}(s) = d\Gamma^{\rm unpol.}(q_1 \ldots q_n)\,\Big(1 + s \cdot h(q_1 \ldots q_n)\Big),$$ where $h \cdot p = 0$.

On the other hand, using KS spinors, we calculate spin amplitudes $\mathcal{N}_\mu$ for decay process. Remembering the textbook definition of spin density matrix we can write:

$$d\Gamma^{\rm pol.}_{KS}(s) = \sum_{\mu,\bar{\mu}=\pm 1/2} \rho_{\mu\bar{\mu}} \mathcal{N}_\mu \mathcal{N}^*_{\bar{\mu}}\, d\Phi_{\rm dec.}$$ where ρ is the spin density matrix.

HOW TO RELATE $\mathcal{N}_\mu$ AND h_μ IN THE TWO CALCULATION METHODS?

Seems easy: from textbook relation $\rho = \sum_{k=0}^{3} \sigma^k\, \hat{s}^k$ where $\hat{s} = (1, \vec{s})$ we get:
$d\Gamma^{\rm pol.}(s) = \hat{s}^a \left[\sigma^a_{\mu\bar{\mu}} \mathcal{N}_\mu \mathcal{N}^*_{\bar{\mu}}\right] d\Phi_{\rm dec.} = \hat{s}^a\, \hat{h}_a\, d\Gamma^{\rm unpol.}$ and therefore we IDENTIFY:

$$\hat{h}_a = (1, \vec{h}(q_1 \ldots q_n)) = \sum_{\rho\bar{\rho}} \sigma^a_{\rho\bar{\rho}} \mathcal{N}_\rho \mathcal{N}^*_{\bar{\rho}} \Big/ \Big(\sum_{\rho} \mathcal{N}_\rho \mathcal{N}^*_\rho\Big)$$ where σ^a are Pauli matrices.

All the above was in the τ rest frame. **WHICH FRAME**?

ONLY in the GPS rest frame! Why? In other than GPS frame we would be forced to replace Pauli matrices with something else; **i.e. with non-standard analog of Clebsch-Gordan coefficients, because the phase relation $(J_x + iJ_y)|-\rangle = |+\rangle$ holds for our spin states only in the GPS frame (note that Pauli σ's in $\rho = \sum_{k=0}^{3} \sigma^k\, \hat{s}^k$ are here just Clebsch-Gordan coefficients for $D^{1/2} \otimes D^{1/2}$).**

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GPS at work: Wigner/Wick rotation

For the τ production and decay process
$e^-(p_1,\lambda_1)+e^+(p_2,\lambda_2)\to\tau^-(q_1,\mu_1)+\tau^+(q_2,\mu_2)$, $\tau^\pm\to X^\pm$,
following the same lines, we may write the spin amplitudes:

$$\mathcal{M}_{\lambda_1\lambda_2}=\sum_{\mu_1\mu_2}\mathcal{M}_{\lambda_1\lambda_2\mu_1\mu_2}\mathcal{N}_{\mu_1}\mathcal{N}'_{\mu_2}$$

and the polarized differential cross-section:

$$d\sigma=\sum_{\mu_i\bar{\mu}_i\lambda_i\bar{\lambda}_i}\rho_{\lambda_1\bar{\lambda}_1}\rho'_{\lambda_2\bar{\lambda}_2}\mathcal{M}_{\lambda_1\lambda_2\mu_1\mu_2}\mathcal{M}^*_{\bar{\lambda}_1\bar{\lambda}_2\bar{\mu}_1\bar{\mu}_2}d\Phi_{\rm prod.}\,\mathcal{N}_{\mu_1}\mathcal{N}^*_{\bar{\mu}_1}d\Phi_\tau\mathcal{N}'_{\mu_2}\mathcal{N}'^*_{\bar{\mu}_2}d\Phi'_\tau$$

In terms of beam polarizations $\hat{s}$ and decay polarimeter vectors $\hat{h}$ we have:

$$d\sigma=\sum_{abcd}\hat{s}^a\hat{s}'^b\;R^{cd}_{ab}\;d\sigma^{\rm prod.}_{\rm unpol.}\;\hat{h}_c\;d\Gamma_{\rm unpol.}\;\hat{h}'_d\;d\Gamma'_{\rm unpol.}$$

where

$$R^{cd}_{ab}=\frac{\sum_{\mu_i\bar{\mu}_i\lambda_i\bar{\lambda}_i}\sigma^a_{\lambda_1\bar{\lambda}_1}\sigma^b_{\lambda_2\bar{\lambda}_2}\mathcal{M}_{\lambda_1\lambda_2\mu_1\mu_2}\mathcal{M}^*_{\bar{\lambda}_1\bar{\lambda}_2\bar{\mu}_1\bar{\mu}_2}\sigma^c_{\bar{\mu}_1\mu_1}\sigma^d_{\bar{\mu}_2\mu_2}}{\sum_{\mu_i\bar{\mu}_i\lambda_i\bar{\lambda}_i}|\mathcal{M}_{\lambda_1\lambda_2\mu_1\mu_2}|^2}$$

The above Wigner-Jacob-Wick spin technology is used in KORALB MC. If the production amplitudes $\mathcal{M}_{\lambda_1\lambda_2\mu_1\mu_2}$ are calculated using KS spinors then we may continue to use the above "spinology" **only if we apply it in the GPS** rest frames of all four fermions!

Consequently, we need Wigner/Wick rotations. Why? See next slide.

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GPS at work: Wigner/Wick rotation

For $e^\pm$ beams we know polarization vectors $\vec{s}_\pm$ in certain well defined rest frames $\mathrm{BRF}_\pm$ of the $e^\pm$ (from the machine people). In other words, we know precisely the Lorentz transformations

$$\boxed{L_\pm^{\rm BRF}:\ \mathrm{BRF}_\pm \longrightarrow \mathrm{LAB}}.$$

On the other hand, from GPS rules, we know transformations

$$\boxed{L_\pm^{\rm GPS}:\ \mathrm{GPS}(e^\pm) \longrightarrow \mathrm{LAB}}.$$

If we use KS spinors for the scattering spin amplitudes, then we have to apply the Wigner/Wick rotation

$\boxed{R_\pm^W = L_\pm^{\rm GPS}(L_\pm^{\rm BRF})^{-1}}$, i.e. rotate $\vec{s}_\pm$ from $\mathrm{BRF}_\pm$ to $\mathrm{GPS}(e^\pm)$.

(In practice this is a rotation around the beam axis).

For $\tau^\pm$ we avoid explicit Wigner/Wick rotation, if the decays of polarized τ's are done (simulated in MC) precisely in $\mathrm{GPS}(\tau^\pm)$ frames.

We may need Wigner/Wick rotation if we want to compare spin amplitudes from KS method and some other method. Next slide shows such a comparison for the spin correlation tensor R_{00}^{cd} from two methods: (1) KS spinors, (2) with Jacob-Wick (JW) helicity states (Jadach Wąs 1984, Tsai 1971); in fact the variant of JW called JW2.

Numerical test of GPS

```
 +++++++++++++++++++++++++++++++++++++++++ Born spin amplitudes ++++++++++++++++++++++++++++++++
     { 1 1 1 1}  { 1 1-1 1}  { 1 1 1-1}  { 1 1-1-1}
     { 1-1 1 1}  { 1-1-1 1}  { 1-1 1-1}  { 1-1-1-1}
     {-1 1 1 1}  {-1 1-1 1}  {-1 1 1-1}  {-1 1-1-1}
     {-1-1 1 1}  {-1-1-1 1}  {-1-1 1-1}  {-1-1-1-1}
[    .000000    .000000] [    .000000    .000000] [    .000000    .000000] [    .000000    .000000]
[    .000000   -.454758] [   -.696227   -.001005] [   -.280666   -.001005] [    .000000    .454758]
[    .000000    .454758] [   -.280666    .001005] [   -.696227    .001005] [    .000000   -.454758]
[    .000000    .000000] [    .000000    .000000] [    .000000    .000000] [    .000000    .000000]
00000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000

####################################### R-matrix GPS fram #######################################
[  1.000000    .000000] [    .000000    .000000] [    .000000    .000000] [    .000000    .000000]
[   .000000    .000000] [   -.023328    .000000] [    .000855    .000000] [    .001871    .000000]
[   .000000    .000000] [   -.000855    .000000] [    .823264    .000000] [   -.386811    .000000]
[   .000000    .000000] [    .001871    .000000] [    .386811    .000000] [   -.153408    .000000]

####################################### R-matrix JacWick  #######################################
[  1.000000    .000000] [    .000000    .000000] [    .000000    .000000] [    .000000    .000000]
[   .000000    .000000] [    .199012    .000000] [    .000000    .000000] [   -.375031    .000000]
[   .000000    .000000] [    .000000    .000000] [    .023418    .000000] [    .000000    .000000]
[   .000000    .000000] [    .375031    .000000] [    .000000    .000000] [   -.824406    .000000]

********************    R analytical      ********************
[    1.000000000000] [      .000000000000] [      .000000000000] [      .000000000000]
[     .000000000000] [      .199012228374] [      .000000000000] [    -.375031393988]
[     .000000000000] [      .000000000000] [      .023418353587] [      .000000000000]
[     .000000000000] [      .375031393988] [      .000000000000] [    -.824406125213]

********************    R Difference      ********************
[     .000000000000] [      .000000000000] [      .000000000000] [      .000000000000]
[     .000000000000] [      .000000000000] [      .000000000000] [      .000000000000]
[     .000000000000] [      .000000000000] [      .000000000000] [      .000000000000]
[     .000000000000] [      .000000000000] [      .000000000000] [      .000000000000]
cos(theta),sin(theta)=      -.884680271149      .466198260229
amfin,sqrt(s)=      1.777000000000     4.000000000000
```

Numerical illustration of Wigner-Wick rotation for one τ-pair production event at $\sqrt{s}$ =4 GeV, $m_\tau = 1.777$ GeV, $\cos\theta = -0.884680271149$. In the upper part we print the complex spin amplitudes $\mathcal{M}^{\rm prod}_{\lambda_1\lambda_2\mu_1\mu_2}$, with the numbering style $\{\lambda_1\lambda_2\mu_1\mu_2\}$ from KS spinors. Then, we show the R^{cd}_{00}, $c, d = 0...3$, correlation matrix as calculated in the GPS frame, and after a (double) Wigner-Wick rotation to JW2 τ rest frames. The result marked "analytical" is obtained analytically (Jadach Wąs 1984) also in the JW2 frames. The difference of the two, in the JW2 frame, is zero.

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Appendix: Wigner-Wick rotations, collection of formulae

STATES in Hilbert space transform under rotation as follows:

$$|\mathring{p},\mu>_R = \sum_{\mu'} |\mathring{p},\mu'> \mathcal{D}^{1/2}_{\mu'\mu}(R), \quad <\mathring{p},\mu|_R = \sum_{\mu'} \mathcal{D}^{1/2\,\dagger}_{\mu\mu'}(R) <\mathring{p},\mu'|.$$

Consequently, SPIN AMPLITUDES transform under four different Wigner rotations:

$$\mathcal{M}_{\lambda_1\lambda_2\mu_1\mu_2} = \sum_{\lambda_i\mu_i} \mathcal{M}_{\lambda'_1\lambda'_2\mu'_1\mu'_2} \mathcal{D}^{1/2}_{\lambda'_1\lambda_1}(R^i_1)\mathcal{D}^{1/2}_{\lambda'_2\lambda_2}(R^i_2)\mathcal{D}^{1/2\,\dagger}_{\mu_1\mu'_1}(R^f_1)\mathcal{D}^{1/2\,\dagger}_{\mu_2\mu'_2}(R^f_2),$$

Both final an initial state spin DENSITY MATRICES transform under Wigner rotation in the same way:

$$(\rho_{\mu\bar{\mu}})_R = \sum_{\mu'\bar{\mu}'} \mathcal{D}^{1/2\,\dagger}_{\mu\mu'}(R)\ \rho_{\mu'\bar{\mu}'}\ \mathcal{D}^{1/2}_{\bar{\mu}'\bar{\mu}}(R)$$

The above transformation induces through $s^k = \mathrm{Tr}(\rho\sigma^k)$ the ordinary rotation transformation of the spin POLARIZATION VECTOR (as it should!) e.g. $\vec{s}$ transforms as a contravariant vector:

$$(s^k)_R = \sum_{k'=1}^{3} R^k{}_{k'}\, s^{k'},$$

On the other hand the h_a POLARIMETER VECTOR (also related to spin amplitudes with Pauli matrices) transforms with transposed rotation matrix, e.g. as a covariant vector:

$$(h_a)_R = \sum_{a'=1}^{3} h_{a'} R^{a'}{}_a$$

Finally the full spin CORRELATION TENSOR transforms as follows:

$$(R^{cd}_{ab})_R = \sum_{a'b'c'd'=1}^{3} R^{c'd'}_{a'b'}\ (R^i_1)^{a'}{}_a\ (R^i_2)^{b'}{}_b\ (R^f_1)^c{}_{c'}\ (R^f_2)^d{}_{d'}.$$

Easy?!

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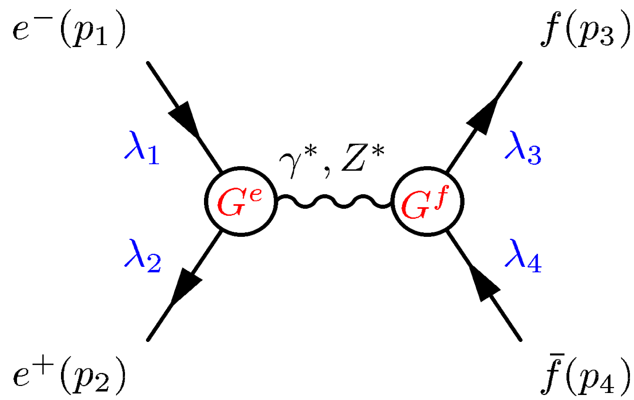
Appendix: Born Spin Amplitudes, definition

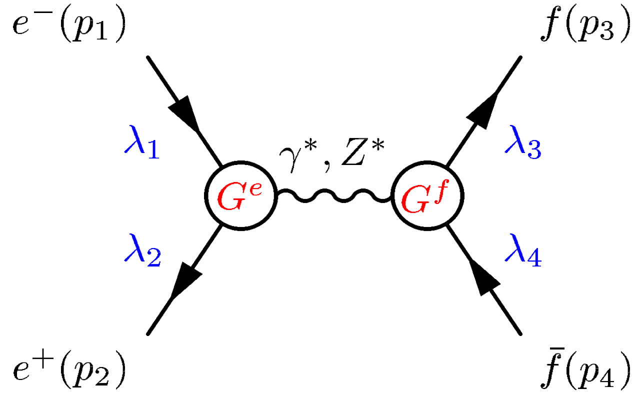


$$\mathfrak{B}\left[{p \atop \lambda}\right](X) = \mathfrak{B}\left[{p_1\, p_2\, p_3\, p_4 \atop \lambda_1\, \lambda_2\, \lambda_3\, \lambda_4}\right](X) =$$

$$= 2ie^2 \sum_{B=\gamma,Z} \frac{\bar{v}(p_2,\lambda_2)\gamma^\mu G^{e,B} u(p_1,\lambda_1)\ \ \bar{u}(p_3,\lambda_3)\gamma_\mu G^{f,B} v(p_4,\lambda_4)}{X^2 - {M_B}^2 + i\Gamma_B X^2/M_B},$$

$$G^{e,B} = \sum_{\lambda=\pm} \frac{1}{2}(1+\lambda\gamma_5) g_\lambda^{e,B}, \quad G^{f,B} = \sum_{\lambda=\pm} \frac{1}{2}(1+\lambda\gamma_5) g_\lambda^{f,B},$$

Appendix: Born Spin Amplitudes in terms of KS spinors



$$\mathfrak{B}\begin{bmatrix} p_1 & p_2 & p_3 & p_4 \\ \lambda_1 & \lambda_2 & \lambda_3 & \lambda_4 \end{bmatrix}(X) = ie^2 \sum_{B=\gamma,Z} \frac{\bar{v}(p_2,\lambda_2)\gamma^\mu G^{e,B} u(p_1,\lambda_1)\ \ \bar{u}(p_3,\lambda_3)\gamma_\mu G^{f,B} v(p_4,\lambda_4)}{X^2 - {M_B}^2 + i\Gamma_B X^2/M_B}$$

$$= 2ie^2 \sum_{B=\gamma,Z} \frac{\delta_{\lambda_1,-\lambda_2}\left[\, g^{e,B}_{\lambda_1} g^{f,B}_{-\lambda_1}\ T_{\lambda_3\lambda_1}\ T'_{\lambda_2\lambda_4} + g^{e,B}_{\lambda_1} g^{f,B}_{\lambda_1}\ U'_{\lambda_3\lambda_2}\ U_{\lambda_1\lambda_4}\right]}{X^2 - {M_B}^2 + i\Gamma_B X^2/M_B},$$

$$T_{\lambda_3\lambda_1} = \bar{u}(p_3,\lambda_3)u(p_1,\lambda_1) = S(p_3,m_3,\lambda_3,p_1,0,\lambda_1),$$

$$T'_{\lambda_2\lambda_4} = \bar{v}(p_2,\lambda_2)v(p_4,\lambda_4) = S(p_2,0,-\lambda_2,p_4,-m_4,-\lambda_4),$$

$$U'_{\lambda_3\lambda_2} = \bar{u}(p_3,\lambda_3)v(p_2,-\lambda_2) = S(p_3,m_3,\lambda_3,p_2,0,\lambda_2),$$

$$U_{\lambda_1\lambda_4} = \bar{u}(p_1,-\lambda_1)v(p_4,\lambda_4) = S(p_1,0,-\lambda_1,p_4,-m_4,-\lambda_4).$$

Conclusions for Part I

- Our primary aim is to use KS spinors for multiple bremsstrahlung spin amplitudes, for massive fermions.
- We would like, however, to take full advantage of the spin amplitudes, i.e. to be able to introduce spin polarization vectors for any fermion and to interface easily the MC simulating polarized decays for unstable fermions like τ and top quark.
- With the GPS upgrade of the KS technique we are fully armed to implement completely the above ambitious scenario!

Outline of Part II

- **Photon polarization vector.**
- **Building blocks for bremsstrahlung amplitudes: U and V matrices.**
- **ISR 1-photon real using KS, isolate IR and non-IR parts, notation.**
- **ISR 1-photon virtual, collect old results.**
- **Coherent Exclusive Exponentiation, zero order $\mathcal{O}(\alpha^0)_{\rm CEEX}$.**
- **Coherent Exclusive Exponentiation, first order $\mathcal{O}(\alpha^1)_{\rm CEEX}$.**
- **Unpolarized $\mathcal{O}(\alpha^r)_{\rm CEEX}$ differential cross section, IR cancellations.**
- **Polarized $\mathcal{O}(\alpha^r)_{\rm CEEX}$ differential x-section, including fermion decays.**
- **CPU time considerations and photon spin randomization.**
- **Preliminary numerical results from $\mathcal{KK}$ Monte Carlo.**

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Photon polarization vector

For a circularly polarized photon with four-momentum k and helicity $\sigma = \pm 1$ we adopt the choice of Kleiss-Stirling and/or Beijing group:

$$(\epsilon^\mu_\sigma(k,\beta))^* = \frac{\bar{u}_\sigma(k)\gamma^\mu u_\sigma(\beta)}{\sqrt{2}\,\bar{u}_{-\sigma}(k)u_\sigma(\beta)}, \quad (\epsilon^\mu_\sigma(k,\zeta))^* = \frac{\bar{u}_\sigma(k)\gamma^\mu \mathfrak{u}_\sigma(\zeta)}{\sqrt{2}\,\bar{u}_{-\sigma}(k)\mathfrak{u}_\sigma(\zeta)},$$

where β is an arbitrary light-like four-vector $\beta^2 = 0$ (axial gauge).
The second choice with $\mathfrak{u}_\sigma(k,\zeta)$ seem to be ours (not exploited by KS).
Using the Chisholm identity:

$$\bar{u}_\sigma(k)\gamma_\mu u_\sigma(\beta)\;\gamma^\mu = 2u_\sigma(\beta)\;\bar{u}_\sigma(k) + 2u_{-\sigma}(k)\;\bar{u}_{-\sigma}(\beta),$$
$$\bar{u}_\sigma(k)\gamma_\mu \mathfrak{u}_\sigma(\zeta)\;\gamma^\mu = 2\mathfrak{u}_\sigma(\zeta)\;\bar{u}_\sigma(k) - 2u_{-\sigma}(k)\;\bar{\mathfrak{u}}_{-\sigma}(\zeta)$$

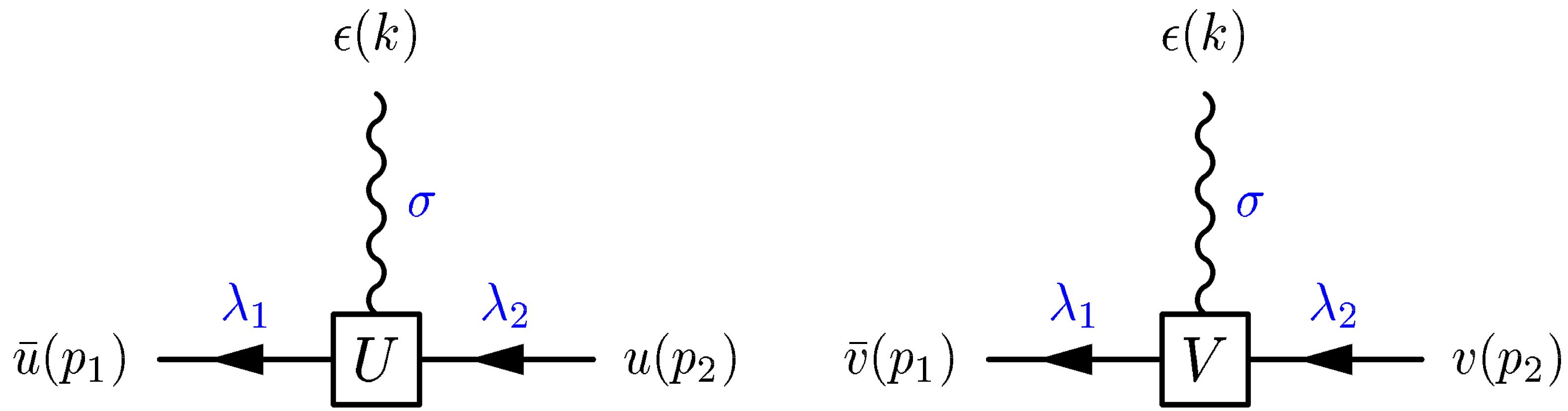
we get useful equivalent expressions:

$$(\not\epsilon_\sigma(k,\beta))^* = \frac{\sqrt{2}\,[u_\sigma(\beta)\bar{u}_\sigma(k) + u_{-\sigma}(k)\bar{u}_{-\sigma}(\beta)]}{\bar{u}_{-\sigma}(k)u_\sigma(\beta)},$$

$$(\not\epsilon_\sigma(k,\zeta))^* = \frac{\sqrt{2}\,[\mathfrak{u}_\sigma(\zeta)\bar{u}_\sigma(k) - u_{-\sigma}(k)\bar{\mathfrak{u}}_{-\sigma}(\zeta)]}{\sqrt{2\zeta k}}.$$

Building blocks for bremsstrahlung amplitudes: U and V matrices.

In the calculation of the bremsstrahlung amplitudes we shall use the following basic building blocks:



$$U \binom{k}{\sigma} \begin{bmatrix} p_1 & p_2 \\ \lambda_1 & \lambda_2 \end{bmatrix} = U^{\sigma}{}_{\lambda_1,\lambda_2}(k,p_1,m_1,p_2,m_2) = \bar{u}(p_1,\lambda_1)\ \not{\epsilon}^{\star}_{\sigma}(k)\ u(p_2,\lambda_2),$$

$$V \binom{k}{\sigma} \begin{bmatrix} p_1 & p_2 \\ \lambda_1 & \lambda_2 \end{bmatrix} = V^{\sigma}{}_{\lambda_1,\lambda_2}(k,p_1,m_1,p_2,m_2) = \bar{v}(p_1,\lambda_1)\ \not{\epsilon}^{\star}_{\sigma}(k)\ v(p_2,\lambda_2).$$

The four-momentum conservation is not assumed in the above vertex-like objects.

U and V transition matrices, analytical formulas

In the case of $\epsilon_\sigma(k,\zeta)$ the transition matrices are rather simple:

$$U^{+}(k,p_1,m_1,p_2,m_2)=\sqrt{2}\begin{bmatrix} \sqrt{\frac{2\zeta p_1}{2\zeta k}}\, s_+(k,\hat{p_1}), & 0 \\ m_2\sqrt{\frac{2\zeta p_1}{2\zeta p_2}}-m_1\sqrt{\frac{2\zeta p_2}{2\zeta p_1}}, & \sqrt{\frac{2\zeta p_2}{2\zeta k}}\, s_+(k,\hat{p_2}) \end{bmatrix},$$

$$U^{-}_{\lambda_1,\lambda_2}(k,p_1,m_1,p_2,m_2)=\left[-U^{+}_{\lambda_2,\lambda_1}(k,p_2,m_2,p_1,m_1)\right]^*,$$

$$V^{\sigma}_{\lambda_1,\lambda_2}(k,p_1,m_1,p_2,m_2)=U^{\sigma}{}_{-\lambda_1,-\lambda_2}(k,p_1,-m_1,p_2,-m_2).$$

The general case, with $\epsilon_\sigma(k,\beta)$, looks a little bit more complicated:

$$U^{+}(k,p_1,m_1,p_2,m_2)=\frac{\sqrt{2}}{s_-(k,\beta)}\times$$

$$\begin{bmatrix} s_+(\hat{p}_1,k)s_-(\beta,\hat{p}_2)+m_1m_2\sqrt{\frac{2\zeta\beta}{2\zeta p_1}\frac{2\zeta k}{2\zeta p_2}}, & m_1\sqrt{\frac{2\zeta\beta}{2\zeta p_1}}s_+(k,\hat{p}_2)+m_2\sqrt{\frac{2\zeta\beta}{2\zeta p_2}}s_+(\hat{p}_1,k) \\ m_1\sqrt{\frac{2\zeta k}{2\zeta p_1}}s_-(\beta,\hat{p}_2)+m_2\sqrt{\frac{2\zeta k}{2\zeta p_2}}s_-(\hat{p}_1,\beta), & s_-(\hat{p}_1,\beta)s_+(k,\hat{p}_2)+m_1m_2\sqrt{\frac{2\zeta\beta}{2\zeta p_1}\frac{2\zeta k}{2\zeta p_2}} \end{bmatrix}$$

The first expressions is optimized for fast numerical evaluation.
Both are not M-matrices of KS, but rather products of them.

Diagonality property of U and V

For $p_1 = p_2 = p$ the matrices U and V become diagonal:

$$U\binom{k}{\sigma}\begin{bmatrix}p & p\\ \lambda_1 & \lambda_2\end{bmatrix} = V\binom{k}{\sigma}\begin{bmatrix}p & p\\ \lambda_1 & \lambda_2\end{bmatrix} = b_\sigma(k,p)\;\delta_{\lambda_1\lambda_2},$$

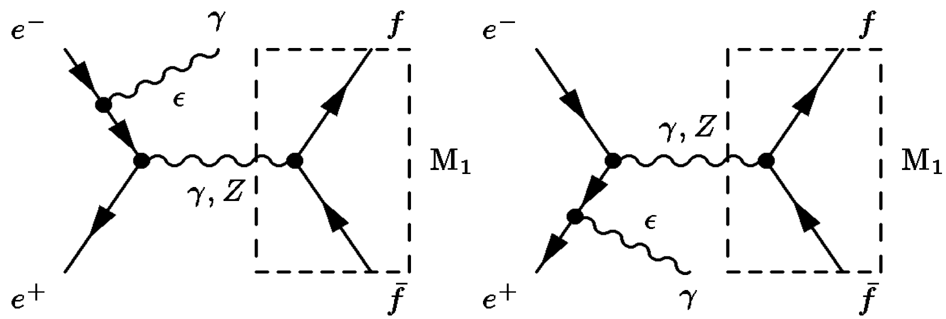
$$b_\sigma(k,p) = \sqrt{2}\,\frac{\bar{u}_\sigma(k)\;\not{p}\;\mathfrak{u}_\sigma(\zeta)}{\bar{u}_{-\sigma}(k)\mathfrak{u}_\sigma(\zeta)} = \sqrt{2}\,\sqrt{\frac{2\zeta p}{2\zeta k}}\,s_\sigma(k,\hat{p}).$$

The above diagonality property also holds in the general case of $\epsilon(k,\beta)$:

$$b_\sigma(k,p) = \frac{\sqrt{2}}{s_{-\sigma}(k,\beta)}\left(s_{-\sigma}(\beta,\hat{p})s_\sigma(\hat{p},k) + \frac{m^2}{2\zeta\hat{p}}\sqrt{(2\beta\zeta)\;(2\zeta k)}\right).$$

Thanks to diagonality we easily obtain/explore the soft limit of the multi-photon amplitudes.

ISR First order real 1-photon



$$\mathcal{M}_1^{\rm ISR}\left({p_1\, p_2\, k \atop \lambda_1\, \lambda_2\, \sigma}\right) = \frac{eQ_e}{2kp_1}\, \bar{v}(p_2,\lambda_2)\, \mathbf{M}_1\, (\not{p}_1 + m - \not{k})\, \not{\epsilon}^\star_\sigma(k)\, u(p_1,\lambda_1)$$
$$+ \frac{eQ_e}{2kp_2}\, \bar{v}(p_2,\lambda_2)\, \not{\epsilon}^\star_\sigma(k)\, (-\not{p}_2 + m + \not{k})\, \mathbf{M}_1\, u(p_1,\lambda_1)$$

Using spinor completeness relation for $\not{p} \pm m$ parts we shall now express the above amplitudes in terms of U and V matrices; separately IR terms $\sim \not{p} \pm m$ and finite terms $\sim \not{k}$. See next slide.

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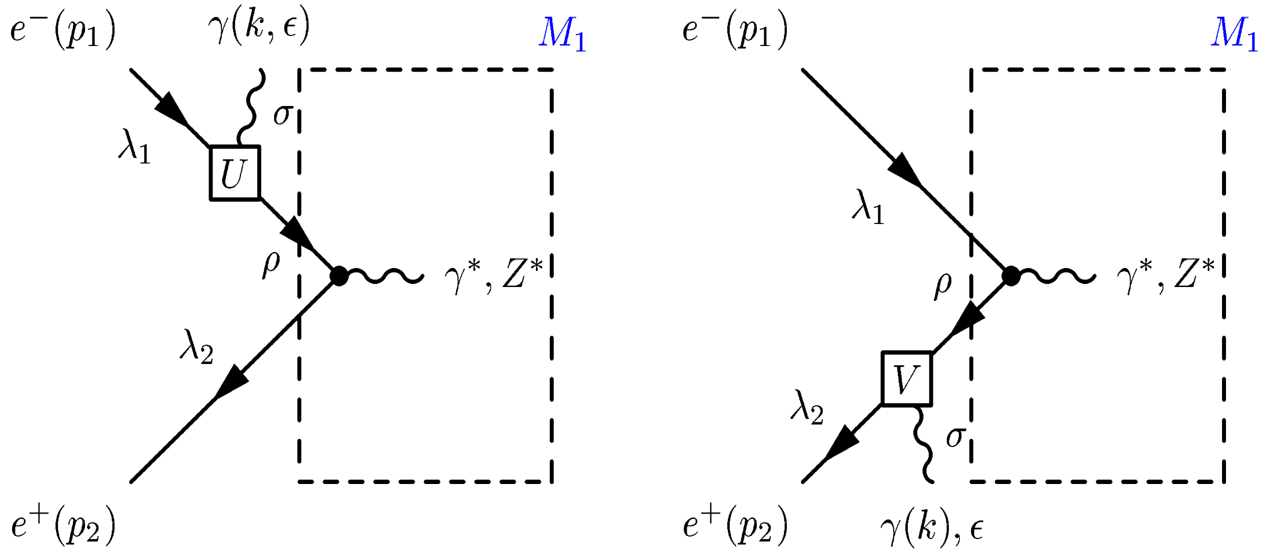
ISR in terms of U and V matrices; IR is separated



$$\mathcal{M}_1^{\rm ISR}\begin{pmatrix} p_1 & p_2 & k \\ \lambda_1 & \lambda_2 & \sigma \end{pmatrix} =$$

$$= \frac{eQ_e}{2kp_1}\sum_\rho \mathfrak{B}_1 \begin{bmatrix} p_1 & p_2 \\ \rho & \lambda_2 \end{bmatrix} U \begin{pmatrix} k \\ \sigma \end{pmatrix} \begin{bmatrix} p_1 & p_1 \\ \rho & \lambda_1 \end{bmatrix} - \frac{eQ_e}{2kp_2}\sum_\rho V \begin{pmatrix} k \\ \sigma \end{pmatrix} \begin{bmatrix} p_2 & p_2 \\ \lambda_2 & \rho \end{bmatrix} \mathfrak{B}_1 \begin{bmatrix} p_1 & p_2 \\ \lambda_1 & \rho \end{bmatrix}$$

$$- \frac{eQ_e}{2kp_1}\sum_\rho \mathfrak{B}_1 \begin{bmatrix} k & p_2 \\ \rho & \lambda_2 \end{bmatrix} U \begin{pmatrix} k \\ \sigma \end{pmatrix} \begin{bmatrix} k & p_1 \\ \rho & \lambda_1 \end{bmatrix} + \frac{eQ_e}{2kp_2}\sum_\rho V \begin{pmatrix} k \\ \sigma \end{pmatrix} \begin{bmatrix} p_2 & k \\ \lambda_2 & \rho \end{bmatrix} \mathfrak{B}_1 \begin{bmatrix} p_1 & k \\ \lambda_1 & \rho \end{bmatrix}$$

Notation: Final fermion spin indices omitted; $\mathfrak{B}_1 \begin{bmatrix} p_1 & p_2 \\ \lambda_1 & \lambda_2 \end{bmatrix}$ really is $\mathfrak{B} \begin{bmatrix} p_1 & p_2 & p_3 & p_4 \\ \lambda_1 & \lambda_2 & \lambda_3 & \lambda_4 \end{bmatrix}$.

ISR First order, 1-photon, IR and non-IR separately

$$\mathcal{M}_1^{\rm ISR}\binom{p_1\,p_2\,k}{\lambda_1\,\lambda_2\,\sigma} = \mathfrak{s}_\sigma^{(1)}(k)\,\mathfrak{B}_1\begin{bmatrix}p_1\,p_2\\ \lambda_1\,\lambda_2\end{bmatrix} + r^{(1)}\begin{bmatrix}p_1\,p_2\,k\\ \lambda_1\,\lambda_2\,\sigma\end{bmatrix}(k).$$

Not only the soft part proportional to

$$\mathfrak{s}_\sigma^{(1)}(k) = eQ_e\frac{b_\sigma(k,p_1)}{2kp_1} - eQ_e\frac{b_\sigma(k,p_2)}{2kp_2},\quad |\mathfrak{s}_\sigma^{(1)}(k)|^2 = -\frac{e^2Q_e^2}{2}\left(\frac{p_1}{kp_1}-\frac{p_2}{kp_2}\right)^2,$$

is now clearly separated but also the very important remaining non-IR part necessary for exponentiation is obtained:

$$r^{(1)}\begin{bmatrix}p_1\,p_2\,k\\ \lambda_1\,\lambda_2\,\sigma\end{bmatrix}(k) = -\frac{eQ_e}{2kp_1}\sum_\rho \mathfrak{B}_1\begin{bmatrix}k\,p_2\\ \rho\,\lambda_2\end{bmatrix} U\binom{k}{\sigma}\begin{bmatrix}k\,p_1\\ \rho\,\lambda_1\end{bmatrix}$$
$$+\frac{eQ_e}{2kp_2}\sum_\rho V\binom{k}{\sigma}\begin{bmatrix}p_2\,k\\ \lambda_2\,\rho\end{bmatrix}\mathfrak{B}_1\begin{bmatrix}p_1\,k\\ \lambda_1\,\rho\end{bmatrix}.$$

(Final fermion spin indices are omitted in $r^{(1)}$ and $\mathfrak{B}_1$.)

The case of FSR can be discussed/analyzed along the same lines, leading to similar elements: $\mathfrak{s}_\sigma^{(0)}(k)$ and $r^{(0)}\begin{bmatrix}p_3\,p_4\,k\\ \lambda_3\,\lambda_4\,\sigma\end{bmatrix}(k)$.

First order, one virtual photon

The $\mathcal{O}(\alpha^1)$ contribution with 1 virtual and 0 real photons:
$\mathcal{M}_0^{(1)}\left[{p \atop \lambda}\right](X) = \mathfrak{B}\left[{p \atop \lambda}\right](X)\left[1 + Q_e^2 F_1(s, m_\gamma) + Q_f^2 F_1(s, m_\gamma)\right] + \mathcal{M}_{\rm box}\left[{p \atop \lambda}\right](X)$,
where F_1 is the electric formfactor regularized with m_γ. (We omit F_2, this is justified for light final fermions. To be restored later on). In F_1 we already keep exact final fermion mass.

In the present work we use spin amplitudes for γ-γ and γ-Z boxes following Brown, Decker and Paschos (1984):

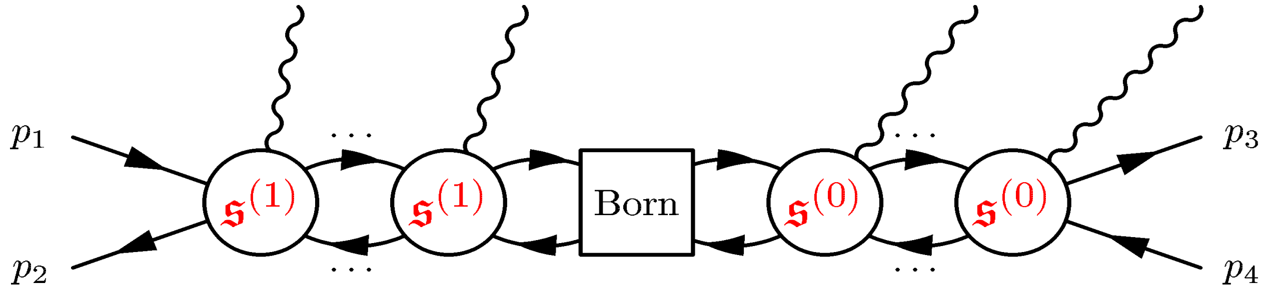
$$\mathcal{M}_{\rm Box}\left[{p \atop \lambda}\right](X) = ie^2 \times$$

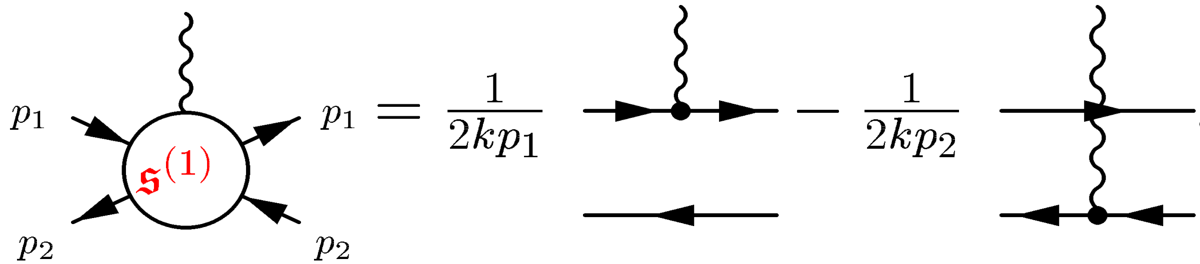
$$\sum_{B=\gamma,Z} \frac{g^{e,B}_{\lambda_1} g^{f,B}_{-\lambda_1}\; T_{\lambda_3\lambda_1} T'_{\lambda_2\lambda_4} + g^{e,B}_{\lambda_1} g^{f,B}_{\lambda_1}\; U'_{\lambda_3\lambda_2} U_{\lambda_1\lambda_4}}{X^2 - {M_B}^2 + i\Gamma_B X^2/M_B}\; \delta_{\lambda_1,-\lambda_2}\delta_{\lambda_3,-\lambda_4}$$

$$\times \frac{\alpha}{\pi} Q_e Q_f \left[\delta_{\lambda_1,\lambda_3}\; f_{\rm BDP}(\bar{M}_B^2, m_\gamma, s, t, u) - \delta_{\lambda_1,-\lambda_3}\; f_{\rm BDP}(\bar{M}_B^2, m_\gamma, s, u, t)\right],$$

where $\bar{M}_Z^2 = M_Z^2 - iM_Z\Gamma_Z$, $\bar{M}_\gamma^2 = m_\gamma^2$. and function $f_{\rm BDP}$ is defined in Brown et.al (1984); Mandelstam variables s, t and u defined as usual.

Coherent Exclusive Exponentiation, zero order $\mathcal{O}(\alpha^0)_{\rm CEEX}$





$$\mathfrak{M}_n^{(0)}\begin{pmatrix} p & k_1 & k_2 & \dots & k_n \\ \lambda & \sigma_1 & \sigma_2 & \dots & \sigma_n \end{pmatrix} =$$

$$= e^{\alpha B_4(p_1,\dots,p_4)} \sum_{\{\wp\}} \frac{X_\wp^2}{(p_3+p_4)^2} \mathfrak{B}\left[{p \atop \lambda}\right](X_\wp)\ \mathfrak{s}^{\wp_1}_{\sigma_1}(k_1)\mathfrak{s}^{\wp_2}_{\sigma_2}(k_2)\dots\mathfrak{s}^{\wp_n}_{\sigma_n}(k_n)$$

Notation: $\left[{p \atop \lambda}\right] \equiv \left[{p_1\, p_2\, p_3\, p_4 \atop \lambda_1\, \lambda_2\, \lambda_3\, \lambda_4}\right]$

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Coherent Exclusive Exponentiation, zero order $\mathcal{O}(\alpha^0)_{\rm CEEX}$

$$\mathfrak{M}_n^{(0)}\begin{pmatrix} p & k_1 & k_2 & \dots & k_n \\ \lambda & \sigma_1 & \sigma_2 & \dots & \sigma_n \end{pmatrix} =$$
$$= e^{\alpha B_4(p_1,\dots,p_4)} \sum_{\{\wp\}} \frac{X_\wp^2}{(p_3+p_4)^2} \mathfrak{B}\left[{}^{p}_{\lambda}\right](X_\wp)\ \ \mathfrak{s}_{\sigma_1}^{\wp_1}(k_1)\mathfrak{s}_{\sigma_2}^{\wp_2}(k_2)\dots\mathfrak{s}_{\sigma_n}^{\wp_n}(k_n)$$

- $\mathfrak{s}_\sigma^\wp(k)$ are IR soft photon ISR/FSR factors, one for each real photon.
- $B_4(p_1,...,p_4)$ is IR virtual YFS form-factor.
- The *coherent sum* is taken over set $\{\wp\}$ of all 2^n partitions.
- The partition $\wp$ is defined as a vector $(\wp_1,\wp_2,\dots,\wp_n)$ where $\wp_i=1$ for ISR and $\wp_i=0$ for FSR photon.
- Set of all 2^n partitions is: $\{\wp\}=\{\ (0,0,0,\dots,0),\ (1,0,0,\dots,0),\ (0,1,0,\dots,0),\ (1,1,0,\dots,0),\ \dots\ (1,1,1,\dots,1)\}$.
- $X_\wp = p_1+p_2-\sum_{i=1}^n \wp_i\, k_i$ = 4-mom. in the resonance propagator.

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IR Yennie-Frautschi-Suura virtual form-factor

We take advantage of the Yennie-Frautschi-Suura (1961) fundamental proof of factorization of all virtual IR corrections in the formfactor

$$B_4(p_1,...,p_4) = Q_e^2 B_2(p_1,p_2) + Q_f^2 B_2(p_3,p_4)$$
$$+ Q_e Q_f B_2(p_1,p_3) + Q_e Q_f B_2(p_2,p_4) - Q_e Q_f B_2(p_1,p_4) - Q_e Q_f B_2(p_2,p_3).$$

$$B_2(p,q) \equiv \int \frac{d^4k}{k^2 - m_\gamma^2 + i\epsilon} \frac{i}{(2\pi)^3} \left(\frac{2p+k}{k^2+2kp+i\epsilon} + \frac{2q-k}{k^2-2kq+i\epsilon} \right)^2 .$$

In the above we assume that IR singularities are regularized with finite photon mass m_γ which enters implicitly into all B_2's and $\mathfrak{s}$-factors (and the real photon phase space integrals, in the following discussion).

The above YFS form-factor does not take the "resonance phase" into account. This is of course corrected by non-IR correction order-by-order, for instance by γ-Z box in $\mathcal{O}(\alpha^1)$. This is however not enough. The $\mathcal{O}(\alpha^1)$ ISR*FSR interference in A_{FB} at Z peak will be still 0.5% instead of 0.05%! We therefore exponentiate the "resonace phase" to infinite order, as it is was done by Greco, Pancheri and Srivastava (1975,1980), see next slide.

Greco-Pancheri-Srivastava form-factor

For very narrow resonances the photon emission in the decay process is separated from the photon emission in the production process by very large time-space distance. The ISR*FSR interference is therefore strongly suppressed, typically by Γ/M factors.

In $\mathcal{KK}$ Monte Carlo, since real photons are generated down to arbitrarily low $k^0_{\min} = \epsilon\sqrt{s}/2 \ll \Gamma$, the suppression of **real** photons due to narrow resonance, is realized *automatically*, including all interference effects.

For **virtual** photons we sum up certain subset of the ISR*FSR interferences to infinite order following Greco et al. In practice the rule is: Multiply each part of the spin amplitude proportional to Z-propagator by the additional factor $\boxed{\exp(\delta_G(s,t,u))}$ where:

$$\delta_G(s,t,u) = -2Q_eQ_f\frac{\alpha}{\pi}\ln\left(\frac{t}{u}\right)\ln\left(\frac{M_Z^2 - iM_Z\Gamma_Z - s}{M_Z^2}\right)$$

In $\mathcal{O}(\alpha^1)$ the above exponential factor induces the additional subtraction in the γ-Z box:

$$\mathcal{M}_{\rm box}(s,t,u) \to \mathcal{M}_{\rm box}(s,t,u) - \delta_G(s,t,u)$$

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Simplification (1): Neglect variation of resonance complex phase

$X_\wp \to P = p_3 + p_4$ or $X_\wp = P' = p_1 + p_2$ leads to:

$$\sum_{\{\wp\}} \frac{X_\wp^2}{s'} \mathfrak{B}\left[{}^{p}_{\lambda}\right](X_\wp)\ \mathfrak{s}^{\wp_1}_{\sigma_1}(k_1)\mathfrak{s}^{\wp_2}_{\sigma_2}(k_2)\dots\mathfrak{s}^{\wp_n}_{\sigma_n}(k_n) \longrightarrow$$

$$\longrightarrow \mathfrak{B}\left[{}^{p}_{\lambda}\right](P) \sum_{\{\wp\}} \mathfrak{s}^{\wp_1}_{\sigma_1}(k_1)\mathfrak{s}^{\wp_2}_{\sigma_2}(k_2)\dots\mathfrak{s}^{\wp_n}_{\sigma_n}(k_n) =$$

$$= \mathfrak{B}\left[{}^{p}_{\lambda}\right](P) \prod_{i=1}^{n} \left(\mathfrak{s}^{(0)}_{\sigma_1}(k_i) + \mathfrak{s}^{(1)}_{\sigma_1}(k_i)\right)$$

The sum over all set of partitions $\{\wp\}$ drops out!!!
This is the case of BHLUMI and BHWIDE (at least in s-channel).

Note: The effect of the above simplification is not fatal, ie. the neglected feature is re-introduced *automatically* order-by-order. Of course, it is better to have it built-in to infinite order from the start.

Simplification (2): Neglect ISR*FSR interferences

Let us consider $|\mathcal{M}_n^{(0)}|^2$, the spin amplitude squared:

$$\sum_{\{\wp\}} \sum_{\{\wp'\}} \frac{X_\wp^2}{s'} \mathfrak{B} \left[{p \atop \lambda}\right](X_\wp)\; \mathfrak{s}_{\sigma_1}^{\wp_1}(k_1) \dots \mathfrak{s}_{\sigma_n}^{\wp_n}(k_n) \frac{X_{\wp'}^2}{s'} \left(\mathfrak{B} \left[{p \atop \lambda}\right](X_{\wp'})\; \mathfrak{s}_{\sigma_1}^{\wp'_1}(k_1) \dots \mathfrak{s}_{\sigma_n}^{\wp'_n}(k_n) \right)^*$$

Neglecting ISR*FSR interferences implies that terms $\wp \neq \wp'$ drop out. Only single sum over partitions survives:

$$\sum_{\{\wp\}} \left(\frac{X_\wp^2}{s'} \right)^2 |\mathfrak{B} \left[{p \atop \lambda}\right](X_\wp)|^2 \;\; |\mathfrak{s}_{\sigma_1}^{\wp_1}(k_1)|^2 \;\; |\mathfrak{s}_{\sigma_2}^{\wp_2}(k_2)|^2 \dots |\mathfrak{s}_{\sigma_n}^{\wp_n}(k_n)|^2$$

This simplification is done in KORALZ/YFS2. It is a serious limitation, not easy to correct, downgrading the off-Z-resonance predictions.

Note that in the YFS2 event generator the sum over partition is *randomized* i.e. for a given event only one partition is generated in the M.C. algorithm.

None of these two simplifications are made in the new $\mathcal{KK}$ M.C.

Coherent Exclusive Exponentiation, first order $\mathcal{O}(\alpha^1)_{\rm CEEX}$

$$\mathcal{M}_n^{(1)}\begin{pmatrix} p & k_1 & k_2 & \dots & k_n \\ \lambda & \sigma_1 & \sigma_2 & \dots & \sigma_n \end{pmatrix} = e^{\alpha B_4(p_1,\dots,p_4)} \sum_{\{\wp\}} \prod_{i=1}^{n} \mathfrak{s}_{\sigma_i}^{\wp_i}(k_i)$$

$$\times \Bigg(\mathfrak{B}\begin{bmatrix} p \\ \lambda \end{bmatrix}(X_\wp)\left(1+\delta_{Virt}^{(1)}\right) + \mathcal{R}_{\rm Box}\begin{bmatrix} p \\ \lambda \end{bmatrix}(X_\wp) + \sum_{j=1}^{n} \mathcal{R}_1^{(\wp_j)}\begin{bmatrix} p & k_j \\ \lambda & \sigma_j \end{bmatrix}(X_\wp) \Bigg),$$

$$\mathcal{R}_1^{(\omega)}\begin{bmatrix} p & k \\ \lambda & \sigma \end{bmatrix}(X) \equiv \frac{1}{\mathfrak{s}_\sigma^\omega(k)}\left[r^{(\omega)}\begin{bmatrix} p & k \\ \lambda & \sigma \end{bmatrix}(X) + \left(\frac{(p_3+p_4+(1-\omega)k_j)^2}{(p_3+p_4)^2} - 1 \right) \mathfrak{B}\begin{bmatrix} p \\ \lambda \end{bmatrix}(X) \right]$$

- IR virtual B_4 and real soft $\mathfrak{s}$ cancel each other, as usual.
- New $\mathcal{O}(\alpha^1)$ IR-finite: virtual form-factor $\delta_{Virt}^{(1)}$ and box $\mathcal{R}_{\rm Box}$; real hard-photon contribution is $\mathcal{R}_1^{(\omega)}, \omega = 0, 1$. Identified unambiguously by comparing with the exact $\mathcal{O}(\alpha^1)$ distributions for $n_\gamma = 0, 1$.
- The new sum over all real photons with IR-finite $\mathcal{R}_1^{(\omega)}$ appears. With each term of *this sum* one $\mathfrak{s}$ is replaced by certain IR-finite object.

$\mathcal{O}(\alpha^1)_{\rm CEEX}$ virtual IR-finite $\delta^{(1)}_{Virt}$ and $\mathcal{R}_{\rm Box}$

The IR-finite $\delta^{(1)}_{Virt}$ and $\mathcal{R}_{\rm Box}$ are determined *unambiguously* by identifying for $n=0$ the $\mathcal{O}(\alpha^1)_{\rm CEEX}$ spin amplitudes with the corresponding $\mathcal{O}(\alpha^1)$ spin amplitudes, up to terms of $\mathcal{O}(\alpha^1)$.

In this way we obtain:

$$\delta^{(1)}_{Virt}(s) = Q_e^2 F_1(s,m_\gamma) + Q_f^2 F_1(s,m_\gamma) - Q_e^2 \alpha B_2(s,m_\gamma) - Q_f^2 \alpha B_2(s,m_\gamma).$$

The $\mathcal{R}_{\rm Box}$ is obtained from $\mathcal{M}_{\rm Box}$ in a similar way. Result can be expressed by means of the substitution:

$$f_{\rm BDP}(\bar{M}_B^2, m_\gamma, s, t, u) \to f_{\rm BDP}(\bar{M}_B^2, m_\gamma, s, t, u) - f_{\rm IR}(m_\gamma, t, u),$$

where

$$f_{\rm IR}(m_\gamma,t,u) = \frac{2}{\pi} B_2(m_\gamma,t) - \frac{2}{\pi} B_2(m_\gamma,u) = \ln\left(\frac{t}{u}\right) \ln\left(\frac{m_\gamma^2}{\sqrt{tu}}\right) + \frac{1}{2}\ln\left(\frac{t}{u}\right).$$

Unpolarized $\mathcal{O}(\alpha^r)_{\rm CEEX}$ total x-section; M.E.×Ph.Sp. approach!!!

$$\sigma^{(r)} = \sum_{n=0}^{\infty} \frac{1}{n!} \int d\tau_n(p_1+p_2;\ p_3,p_4,\ k_1,\dots,k_n)\ \frac{1}{4} \sum_{\lambda,\sigma_1,\dots,\sigma_n} \left| \mathcal{M}_n^{(r)} \begin{pmatrix} p\, k_1\, k_2 \\ \lambda\, \sigma_1\, \sigma_2 \end{pmatrix} \dots \begin{pmatrix} k_n \\ \sigma_n \end{pmatrix} \right|^2,$$

where the $d\tau_n$ is the Lorentz invariant phase space (LIPS) integration element. The above total cross section is perfectly IR-finite, as can be checked with a little bit of effort by *analytical* partial differentiation with respect the photon mass $\partial\sigma^{(r)}/\partial m_\gamma = 0$. (This method of validating IR-finiteness was noticed by G. Burgers (1989); the classical method of YFS (1961) relies on the techniques of the Melin transform.) Furthermore, the above integral is perfectly implementable in the M.C. Traditionally, the lower boundary on the real soft photons is defined using the energy cut condition $k^0 > \varepsilon\sqrt{s}/2$ in the laboratory frame. (Faster M.C.) Transition to such a traditional IR boundary leads to an additional overall real photon form-factor $\exp(2\alpha\tilde{B}_4(p_1,...,p_4))$ where

$$\begin{aligned}\tilde{B}_4(p_1,...,p_4) &= Q_e^2\tilde{B}_2(p_1,p_2) + Q_f^2\tilde{B}_2(p_3,p_4)\\ &+ Q_eQ_f\tilde{B}_2(p_1,p_3) + Q_eQ_f\tilde{B}_2(p_2,p_4) - Q_eQ_f\tilde{B}_2(p_1,p_4) - Q_eQ_f\tilde{B}_2(p_2,p_3),\end{aligned}$$

$$\tilde{B}_2(p,q) \equiv \int_{k^0<\varepsilon\sqrt{s}/2} \frac{d^3k}{k^0}\ \frac{(-1)}{8\pi^2} \left(\frac{p}{kp} - \frac{q}{kq}\right)^2.$$

Unpolarized $\mathcal{O}(\alpha^r)_{\rm CEEX}$ total cross section, cont.

$$\sigma^{(r)} = \sum_{n=0}^{\infty} \frac{1}{n!} \int_{k_i^0 > \varepsilon\sqrt{s}/2} d\tau_n(p_1+p_2;\; p_3, p_4,\; k_1, \dots, k_n)\; e^{Y(p_1,\dots,p_4)} \frac{1}{4} \sum_{\lambda,\sigma_i=\pm} \left| \mathfrak{M}_n^{(r)} \begin{pmatrix} p\, k_1\, k_2 \dots k_n \\ \lambda\, \sigma_1\, \sigma_2 \dots \sigma_n \end{pmatrix} \right|^2$$

where

$$Y(p_1,...,p_4) = 2\alpha \tilde{B}_4(m_\gamma; p_1,...,p_4) + 2\alpha \Re B_4(m_\gamma; p_1,...,p_4)$$

is the conventional YFS formfactor defined analytically in terms of logs and Spence functions, and

$$\mathfrak{M}_n^{(r)} \begin{pmatrix} p\, k_1\, k_2 \dots k_n \\ \lambda\, \sigma_1\, \sigma_2 \dots \sigma_n \end{pmatrix} = e^{-\alpha B_4} \mathcal{M}_n^{(r)} \begin{pmatrix} p\, k_1\, k_2 \dots k_n \\ \lambda\, \sigma_1\, \sigma_2 \dots \sigma_n \end{pmatrix}$$

is the spin amplitude without IR virtual divergences, no m_γ any more!.

The above fully exclusive differential cross section is already implemented in the Monte Carlo event generator $\mathcal{KK}$.

Polarized $\mathcal{O}(\alpha^r)_{\rm CEEX}$ total cross section

$$\sigma^{(r)} = \sum_{n=0}^{\infty} \frac{1}{n!} \int_{k_i^0 > \varepsilon\sqrt{s}/2} d\tau_n(p_1+p_2;\; p_3,p_4,\; k_1,\dots,k_n)\; e^{Y(p_1,\dots,p_4)} \sum_{\sigma_i} \sum_{a,b,c,d=0}^{3} \sum_{\lambda_i,\bar\lambda_i} \hat\varepsilon_1^a \hat\varepsilon_2^b\; \sigma^a_{\lambda_1\bar\lambda_1} \sigma^b_{\lambda_2\bar\lambda_2} \mathfrak{M}_n^{(r)} \begin{pmatrix} p & k_1 & k_2 & \dots & k_n \\ \lambda & \sigma_1 & \sigma_2 & \dots & \sigma_n \end{pmatrix} \left[\mathfrak{M}_n^{(r)} \begin{pmatrix} p & k_1 & k_2 & \dots & k_n \\ \bar\lambda & \sigma_1 & \sigma_2 & \dots & \sigma_n \end{pmatrix} \right]^\star \sigma^c_{\bar\lambda_3\lambda_3} \sigma^d_{\bar\lambda_4\lambda_4} \hat h_3^c \hat h_4^d$$

- σ^k for $k=1,2,3$ are Pauli matrices and $\sigma^0_{\lambda,\mu} = \delta_{\lambda,\mu}$ is unit matrix.
- $\hat\varepsilon_1^a, \hat\varepsilon_2^b$, $a,b = 1,2,3$ are the components of the conventional spin polarization vectors of $e^\pm$ and $\hat\varepsilon_i^0 \equiv 1$ in $e^\pm$ rest frame $(\hat\varepsilon_i \cdot p_i = m_e)$.
- $\hat h_3^c \hat h_4^d$ are polarimeter vectors of outgoing fermions. They carry spin information to decay processes $(\hat h_i \cdot p_i = m_f)$.
- $\hat\varepsilon_i^a$ and $\hat h_i^a$ are *primarily* defined in the so called GPS frames of the corresponding fermions. (Important for the use of Pauli matrices!)

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CPU considerations: photon spin randomization

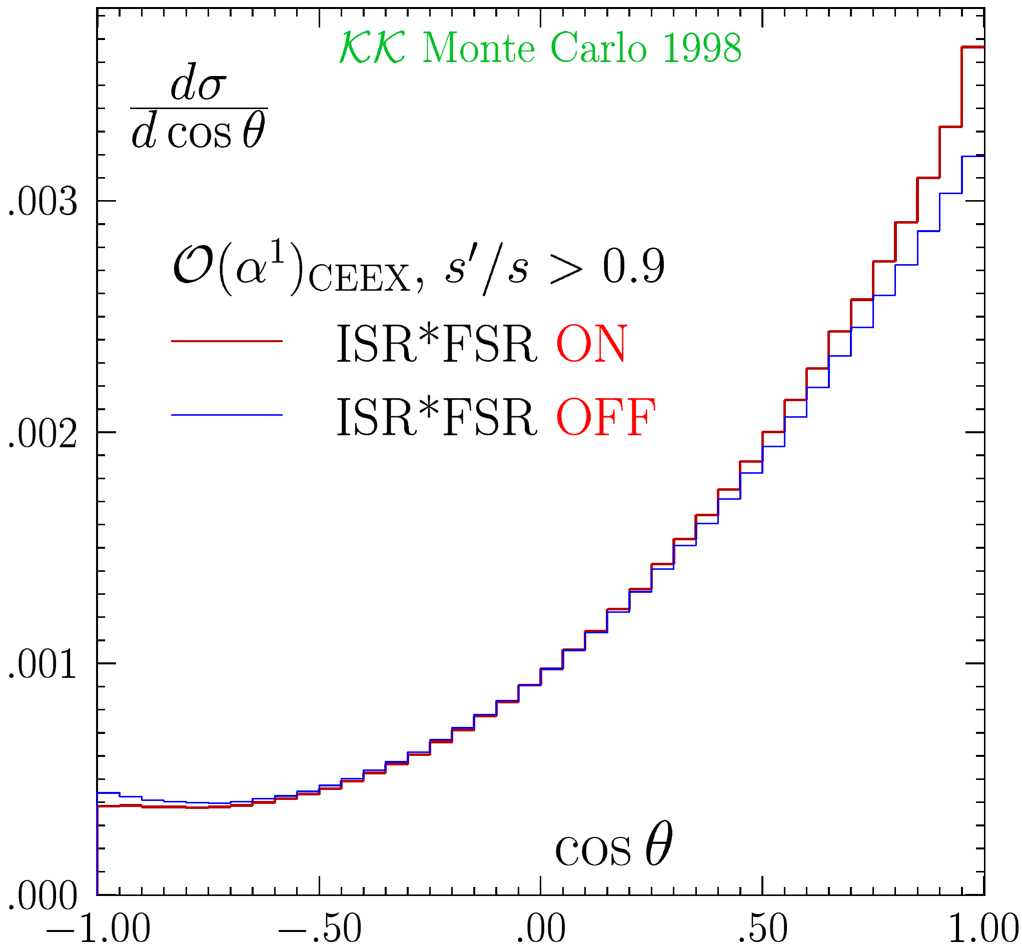
The single spin amplitude $\mathfrak{M}_n^{(1)}$ contains already $2^n(n+1)$ terms (2^n due to ISR/FSR partitions). The grand sum over spins counts $2^n 4^4 4^4 = 2^{n+16}$ terms!!! Altogether we expect up to $N \sim n 2^{2n+16}$ operations in the CPU time expensive complex (16bytes) arithmetics. Typically in $e^- e^+ \to \mu^- \mu^+$ the average photon multiplicity with $k^0 > 1 MeV$ is about 3, corresponding to $N \sim 10^7$ terms. In a sample of 10^4 MC events there will be a couple events with $n = 10$ and $N = 10^{12}$ terms. Partial solutions: $\sum_a \varepsilon_i^a \sigma^a_{\lambda\bar\lambda}$ and the $\mathfrak{s}$-factors evaluated only once, stored and reused (save 2^8).

The trick of *photon spin randomization* speeds up substantially the numerical calculation in the Monte Carlo program: Instead of evaluating the sum over photon spins $\sigma_i, i = 1, ..., n$ we generate randomly one spin sequence of $(\sigma_1, ..., \sigma_n)$ per MC event and the MC weight is calculated only for this particular spin sequence! We save one hefty 2^n factor in the CPU time!

The formal proof of the correctness of this method can be found in Sect. 4 of *"Guide to practical Monte Carlo methods"*, (1998), `http://wwwcn.cern/~jadach`.

ISR*FSR interf. in angular distribution

The influence of ISR*FSR interference on the angular distribution at 189GeV. The cut $s'/s > 0.9$ eliminates on shell Z. This result is totally new from new CEEX exponentiation at the amplitude level.



Interference increases A_{FB} and total x-section about 2%, the effect is located at the very ends of $\cos\theta$ range.

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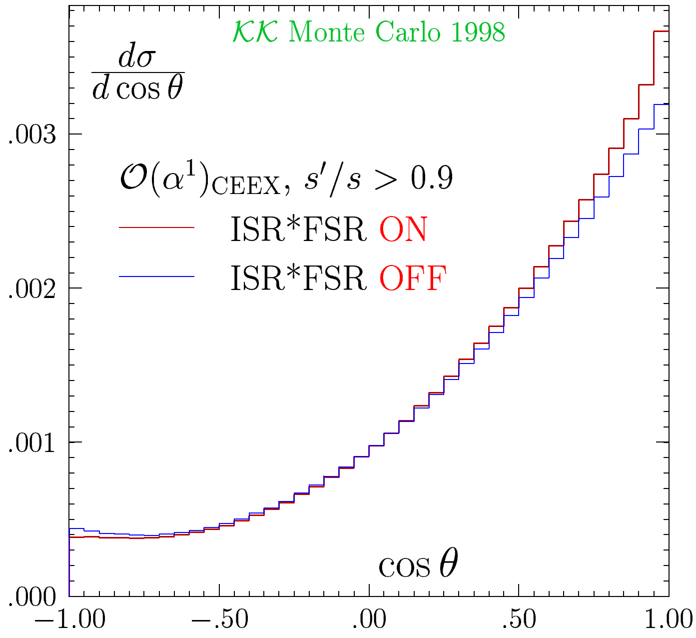
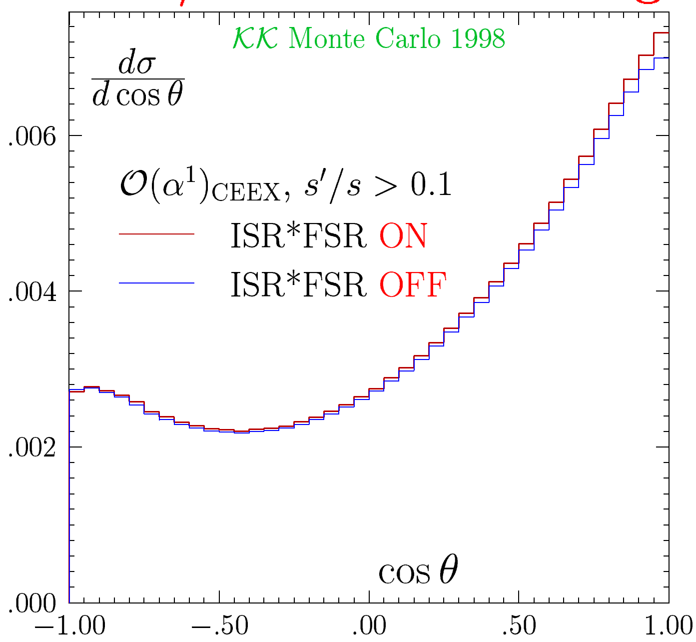
Cut-off dependence at 189GeV

Regular cut-off $s'/s > 0.9$ eliminating Z:



Loose cut-off $s'/s > 0.1$ admitting Z:



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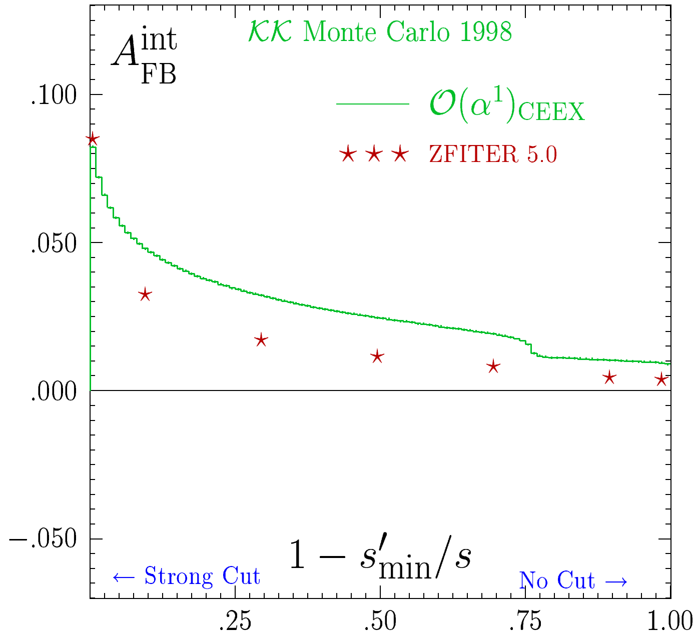
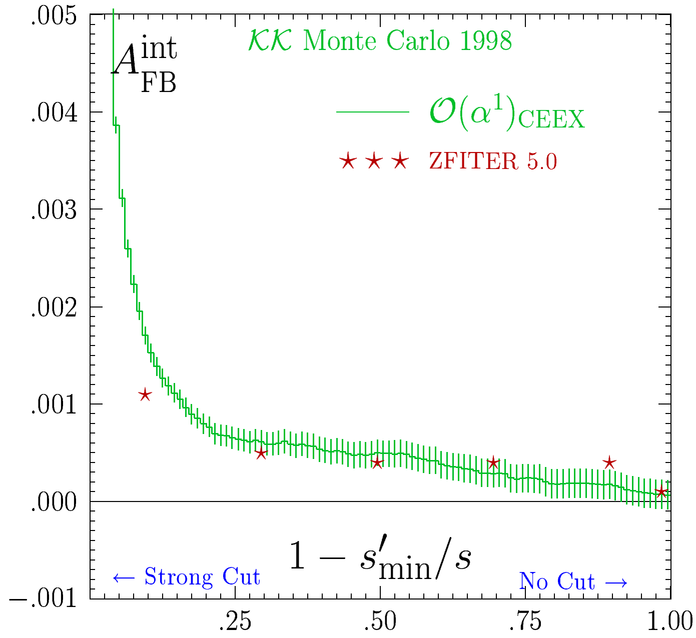
ISR*FSR in A_{FB}, s'-cut dependence, no θ cut

At 189GeV interference in A_{FB} is 2%-5%,:



At Z peak A_{FB} is below 0.1% due to additional exponentiation of virtual correction according to *Greco, Pancheri and Srivastava*



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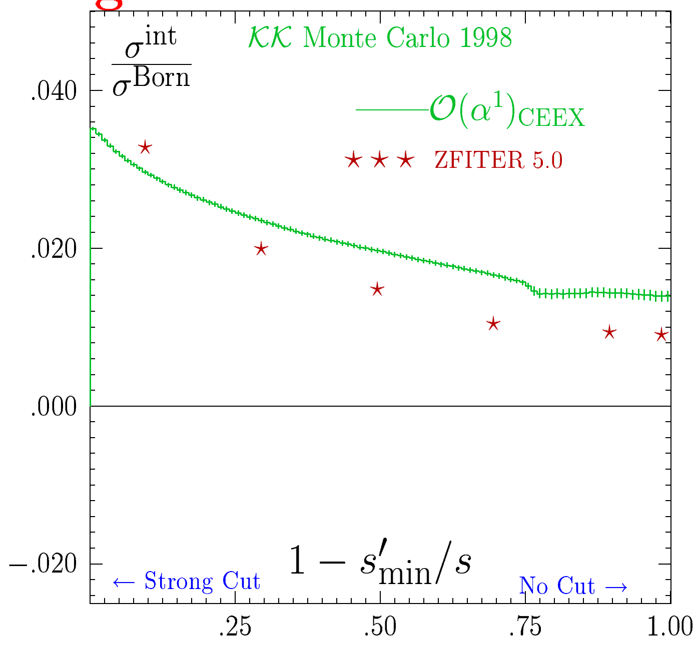
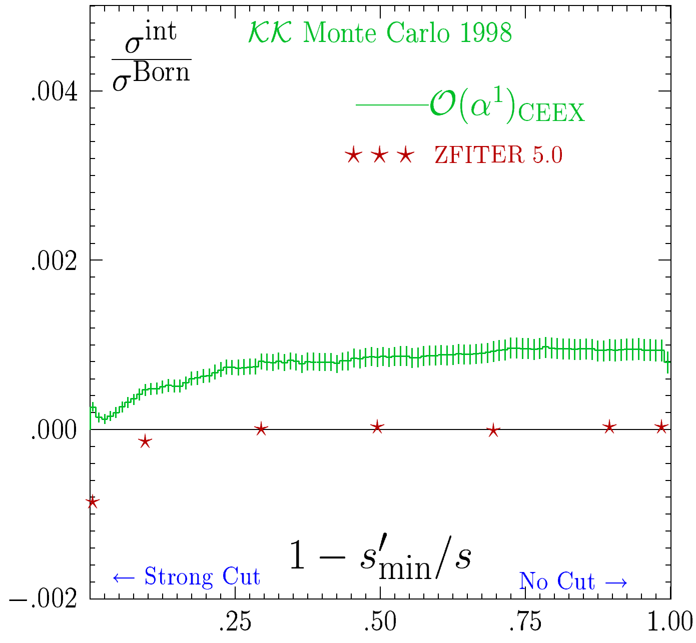
ISR*FSR in σ, cut-off dependence

At 189GeV interference in σ is 2%-5%, full $\cos\theta$ range:



At Z interference is clearly suppressed to below 0.1% or less: again thanks to additional virtual correction exponentiation according to *Greco et al.*



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Comparison with our older MC's

Feature	KORALB	KORALZ	$\mathcal{KK}$ now	$\mathcal{KK}$ 2000
ISR	$\alpha + \alpha L$	$(\alpha + \alpha L + \alpha^2 L^2)_{\rm exp}$	$(\ldots + \alpha^3 L^3)_{\rm exp}$	$(\ldots + \alpha^3 L^3)_{\rm exp}$
FSR	$\alpha + \alpha L$	$(\alpha + \alpha L + \alpha^2 L^2)_{\rm exp}$	$(\ldots + \alpha^2 L^2)_{\rm exp}$	$(\ldots + \alpha^3 L^3)_{\rm exp}$
ISR*FSR int.	$\alpha + \alpha L$	$\alpha + \alpha L$, no exp.	$(\alpha + \alpha L)_{\rm exp}$	$(\alpha + \alpha L)_{\rm exp}$
Exponentiation	NONE	for $\vert M(p_i)\vert^2$	for $M(p_i)$	for $M(p_i)$
El-Weak	No Z-reson.	YES	YES	YES
Beam polar.	long+trans.	longit.	none	long+trans.
τ polar.	long+trans.	longit.	long+trans.	long+trans.
τ decay	yes	yes	yes	yes
Exact m.el. for real photons	up to 1	1, 2coll.	1, 2coll, 3coll.	up to 3
Inclusive mode	—	No	Yes	Yes
$\nu\nu$ channel	—	Yes	No	Yes
ee channel	—	No	No	Yes
tt channel	—	No	No	Yes
WW channel	—	No	No	Yes
beam spread	—	No	Yes	Yes
beamstrahlung	—	No	No	Yes

Conclusions

Clear upgrade path for exclusive exponentiation in QED in the Monte Carlo is established. It is firmly based on spin amplitudes. The main profits are:

- Inclusion of interferences, ISR*FSR.
- All kind of coherence effects, including narrow resonances.
- Complete treatment of spin (also transverse) for beams and final (unstable) fermions.
- Exact M.E. for 2,3 high p_T photons (pending).

First real Monte Carlo implementation is $\mathcal{KK}$ event generator for fermion pair production at LEP, lineacs, μ-colliders, τ and b factories.
Check `http://wwwcn.cern.ch/~jadach`.

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